Spin-bounded correlations: rotation boxes within and beyond quantum theory

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Spin-bounded correlations: rotation boxes within and beyond quantum theory

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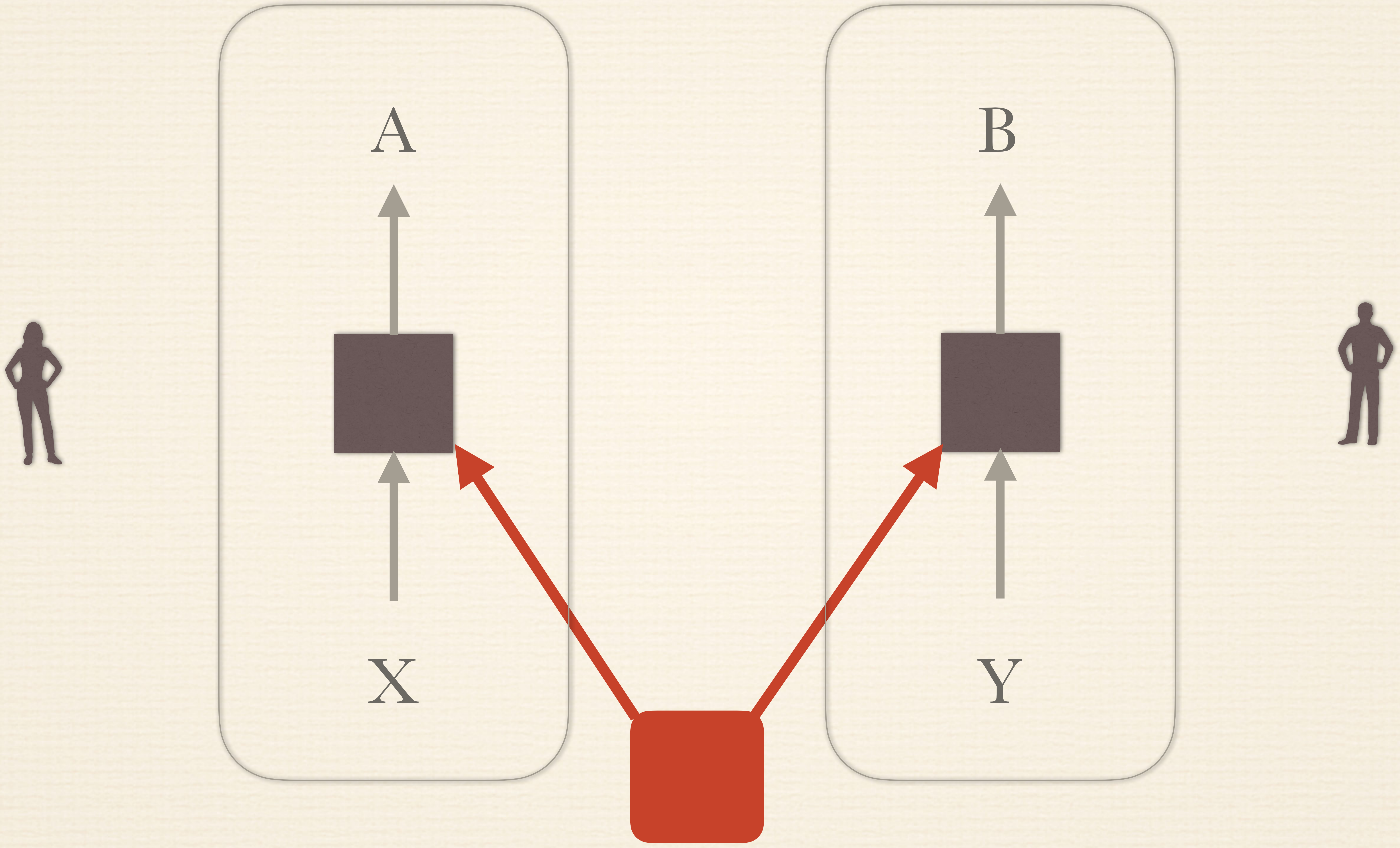
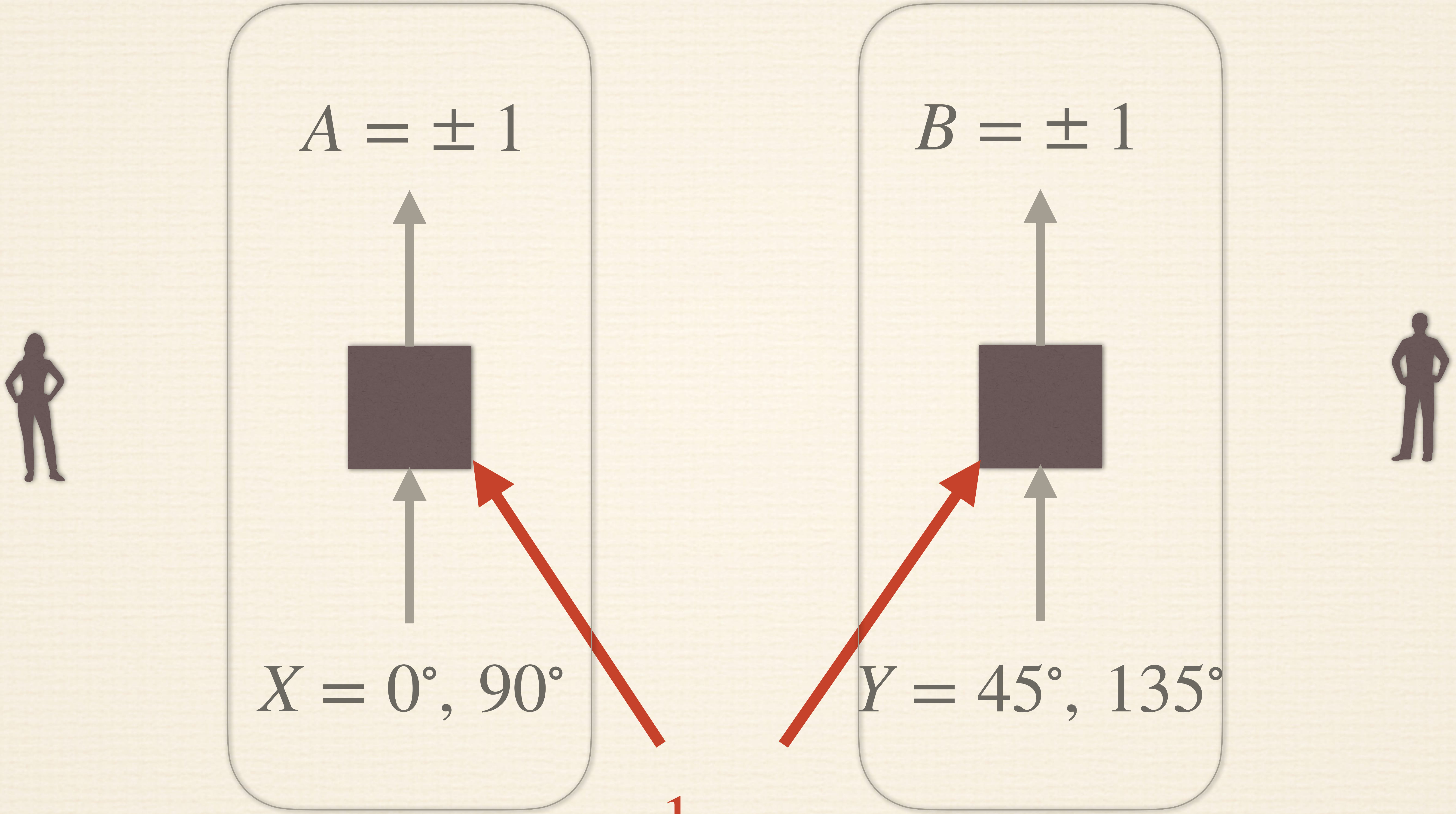
A lesson from Bell's theorem



A lesson from Bell's theorem



$$|\psi\rangle = \frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle)$$

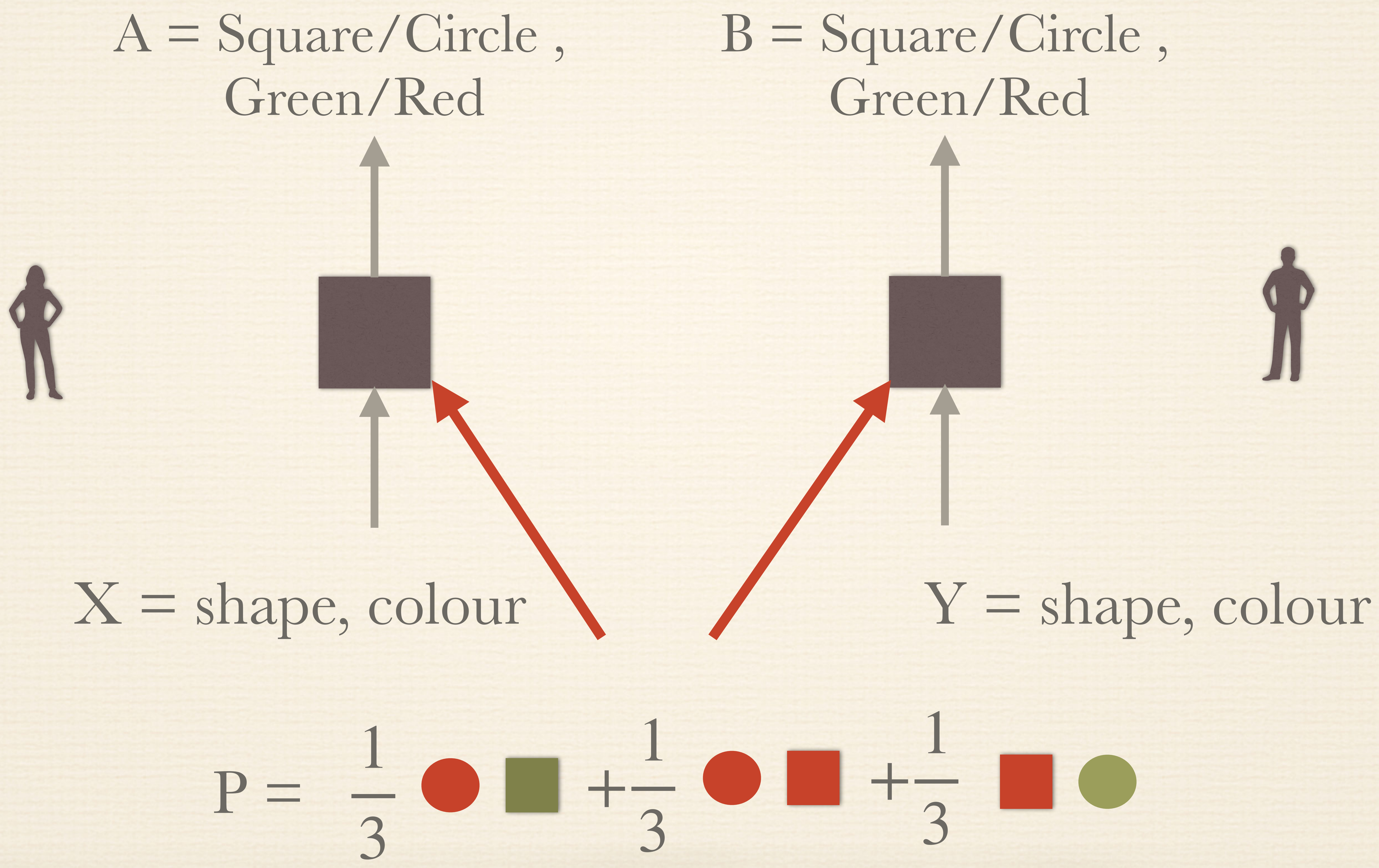
A lesson from Bell's theorem

$$p(A, B \mid X, Y, \psi) = \begin{pmatrix} p(+1, +1 \mid 0^\circ, 45^\circ, \psi) \\ \vdots \\ p(-1, -1 \mid 90^\circ, 135^\circ, \psi) \end{pmatrix} \in \mathbb{R}^{16}$$

$$P_Q := \{p(A, B \mid X, Y, \psi) \mid \psi, X, Y\}$$

Set of correlations for all possible quantum states and 2 outcomes measurements

A lesson from Bell's theorem



A lesson from Bell's theorem

$$p(A,B\,|\,X,Y,P) = \begin{pmatrix} p(G,G\,|\,C,C,P) \\ \vdots \\ p(\square,\square\,|\,S,S,P) \end{pmatrix} \in \mathbb{R}^{16}$$

P_C set of correlations obtained when Alice and Bob share a classical system

$$p(A,B\,|\,X,Y,P) = \sum_{\lambda} p(\lambda\,|\,P)\,p(A\,|\,X,\lambda)\,p(B\,|\,Y,\lambda)$$

A lesson from Bell's theorem

P_C Classical correlations

$$p(A, B \mid X, Y, P) = \sum_{\lambda} p(\lambda \mid P) p(A \mid X, \lambda) p(B \mid Y, \lambda)$$

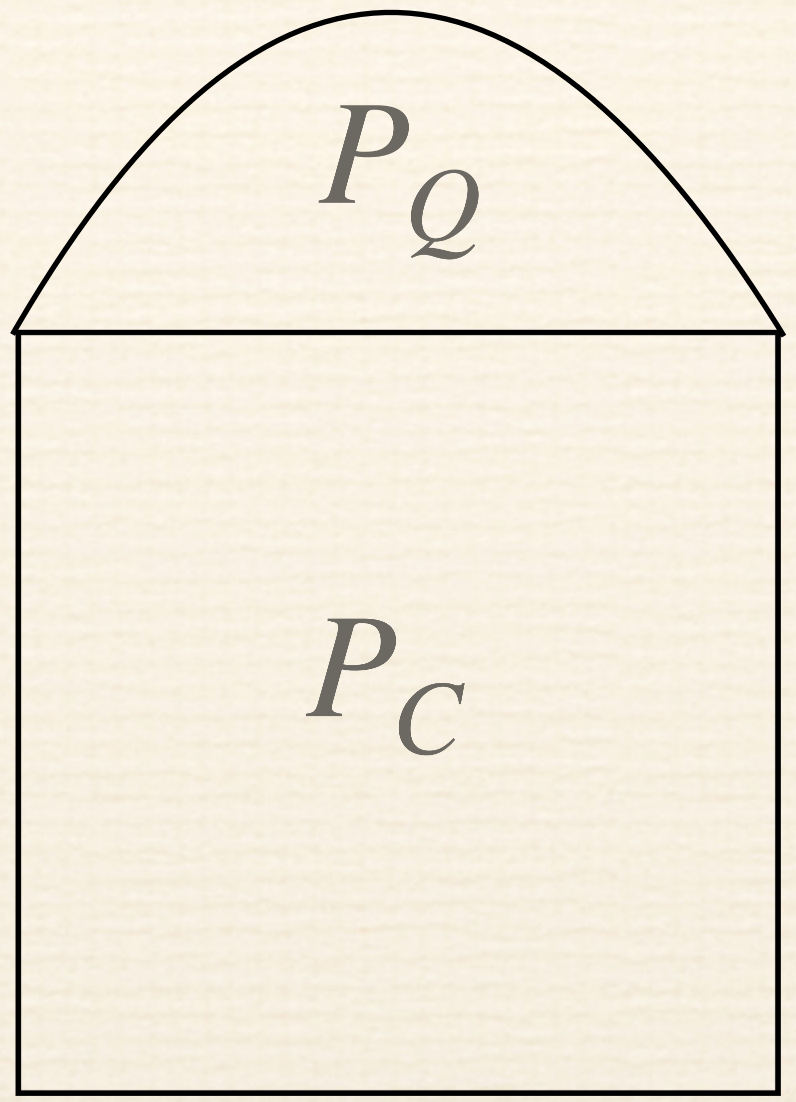
P_Q Quantum correlations

$$p(A, B \mid X, Y, P) = \mathrm{Tr}(\rho_P (X_A \otimes Y_B))$$

A lesson from Bell's theorem

$$P_C \subseteq P_Q$$

Bell's theorem $\qquad P_C \subsetneq P_Q$

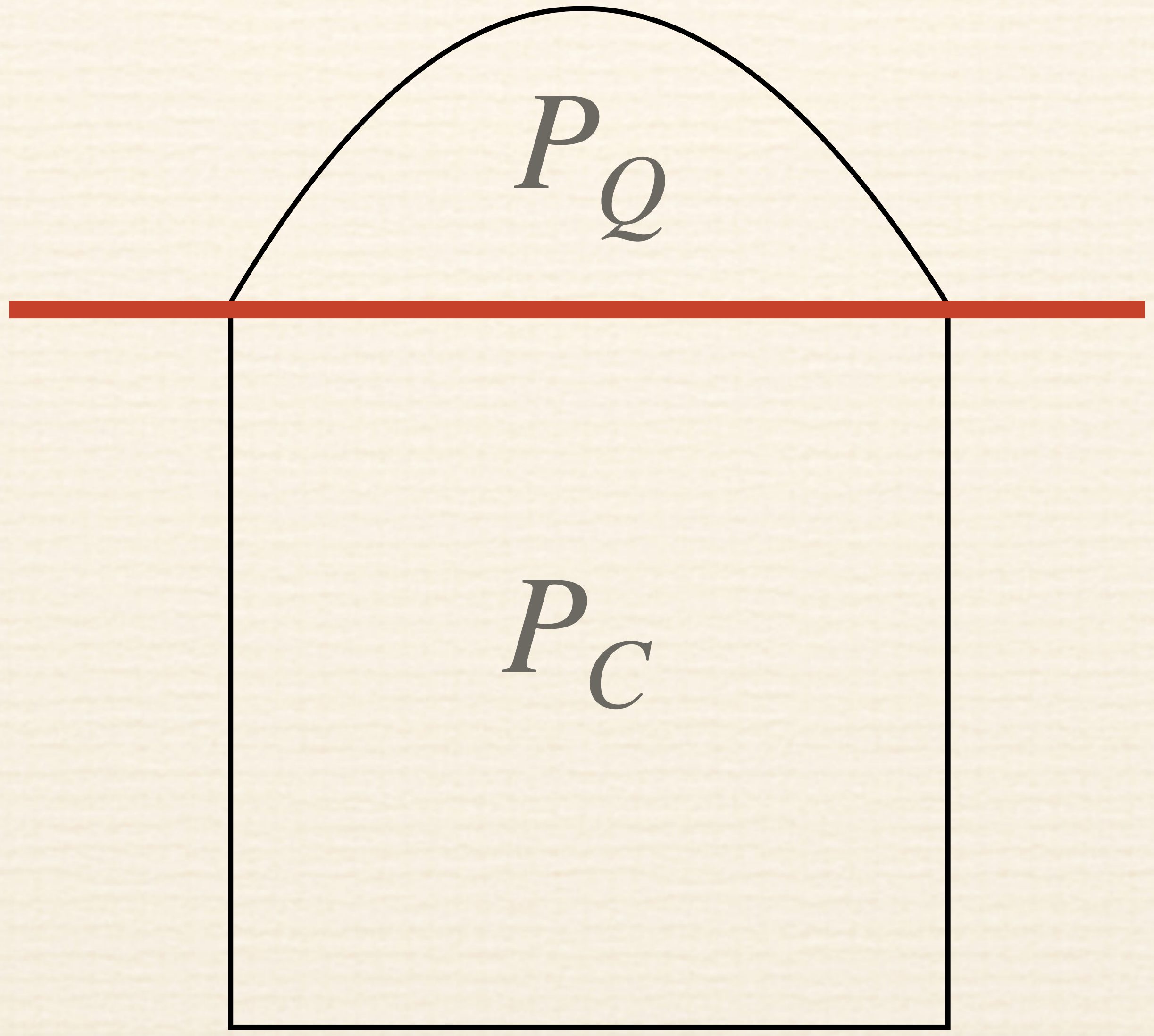


A lesson from Bell's theorem

$$P_C \subseteq P_Q$$

Bell's theorem $\quad P_C \subsetneq P_Q$

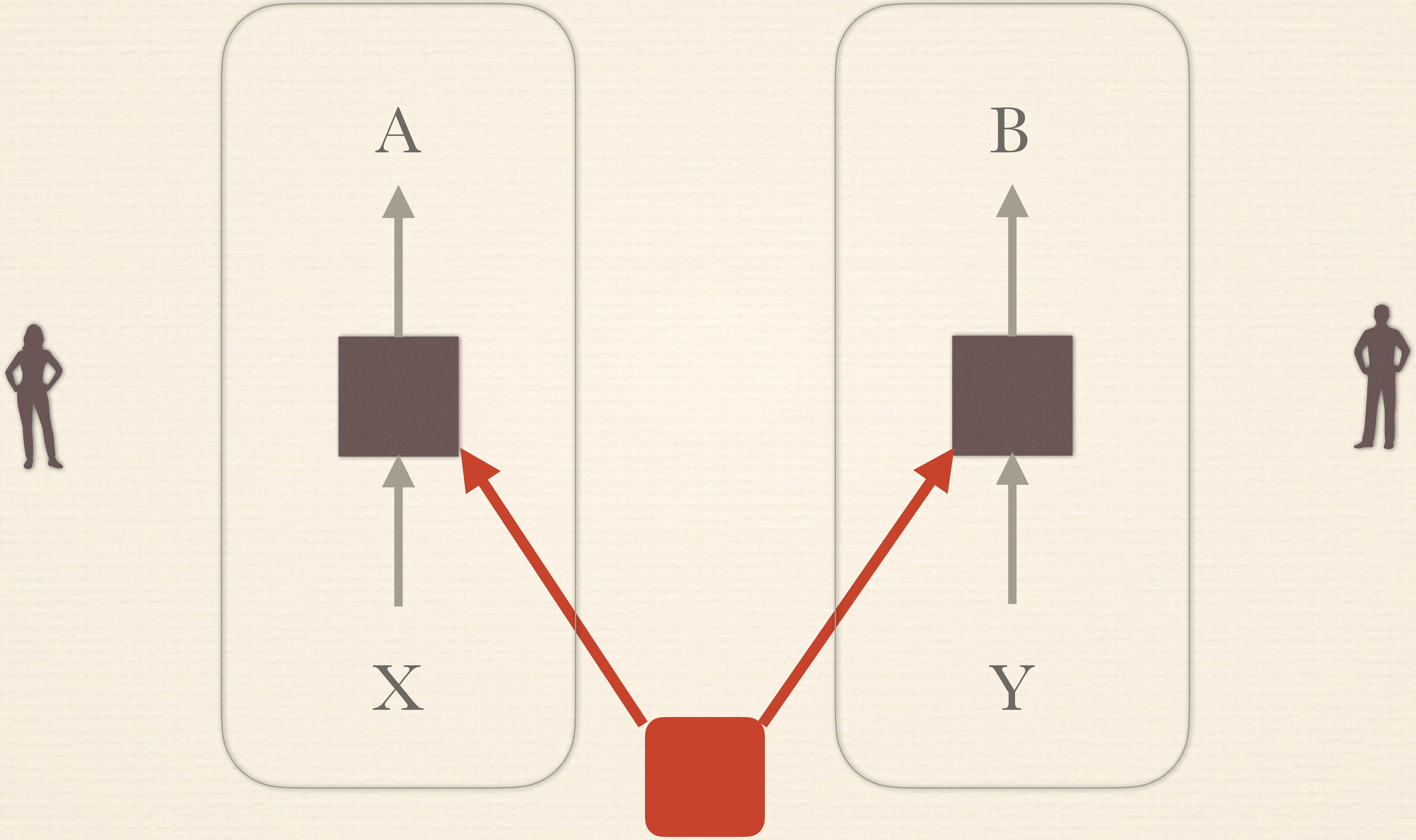
Bell inequality



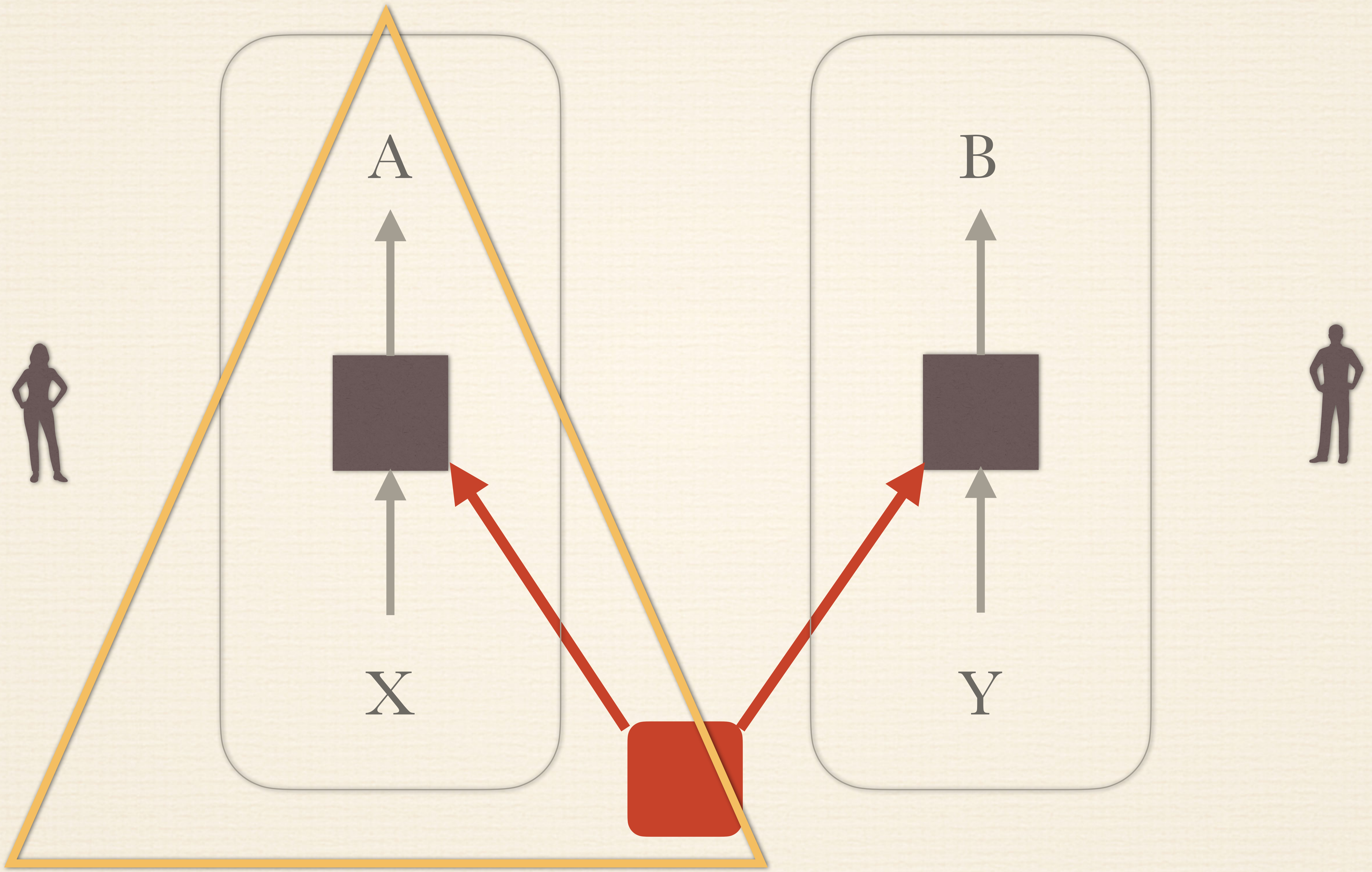
$$\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \leq 2$$

(CHSH)

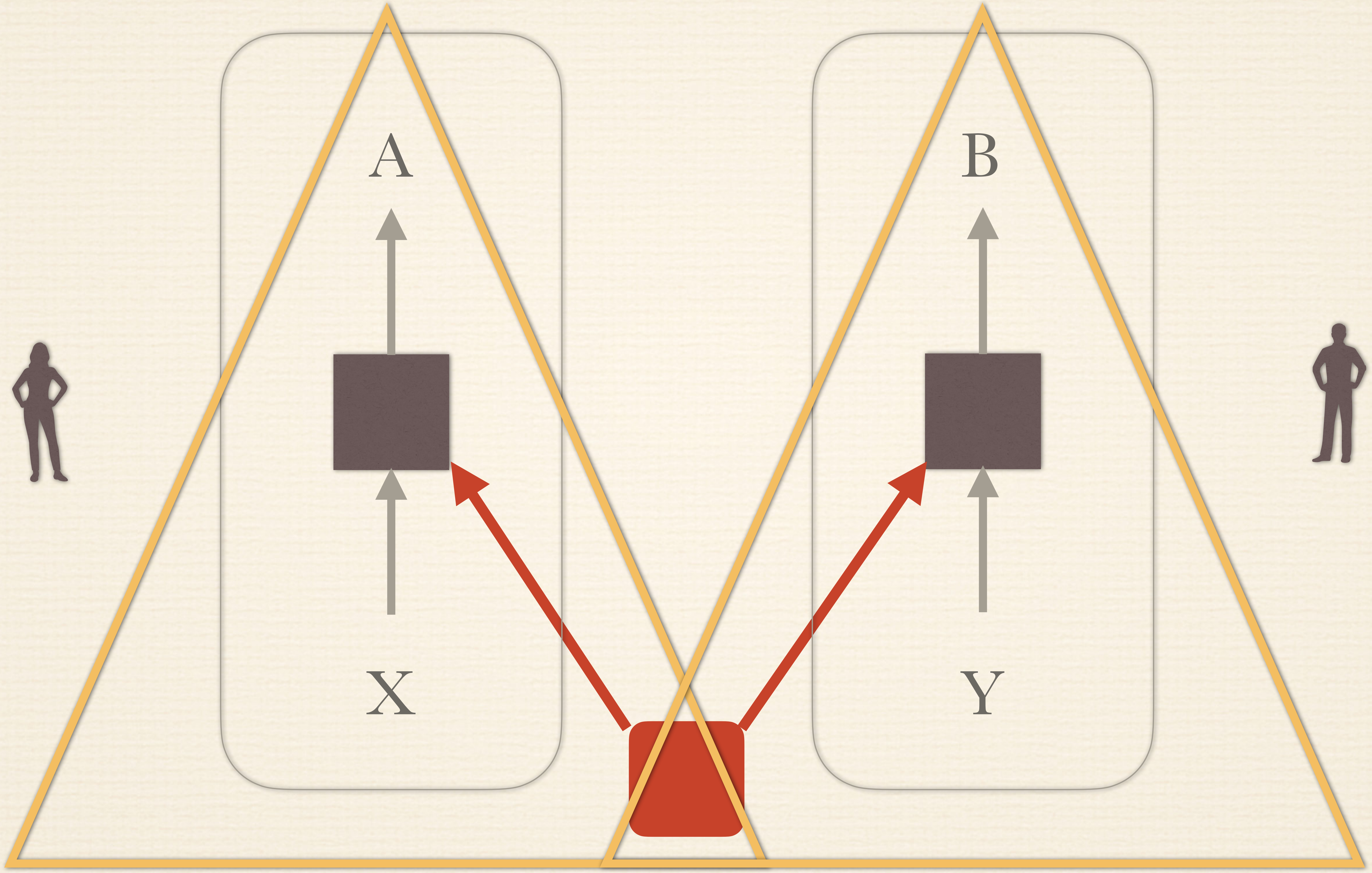
The lesson of Bell's theorem



The lesson of Bell's theorem



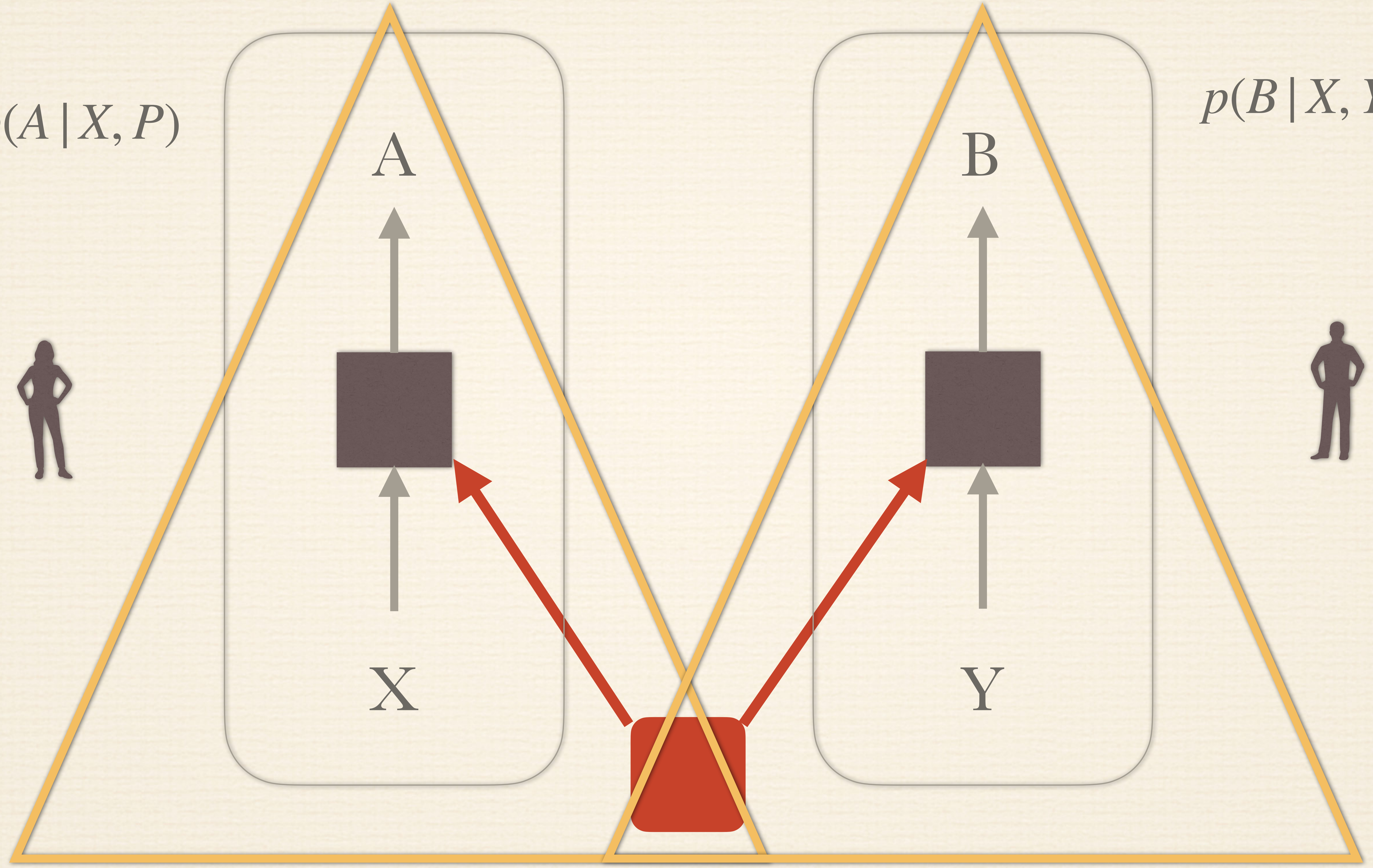
The lesson of Bell's theorem



No-signalling (signal locality)

$$p(A \mid X, Y, P) = p(A \mid X, P)$$

$$p(B \mid X, Y, P) = p(B \mid Y, P)$$



No-signalling

$$p(A|X,Y,P) = p(A|X,P)$$

$$p(B|X,Y,P) = p(B|Y,P)$$

No-signalling is purely operational

It is not the same as locality/ parameter independence

$$p(A|X,Y,\lambda) = p(A|X,\lambda)$$

A model which violates locality still leads to observed statistics which are no-signalling

The space of no-signalling correlations

The set of all correlations which obey the no-signalling condition

$$p(A|X,Y,P) = p(A|X,P)$$

$$p(B|X,Y,P) = p(B|Y,P)$$

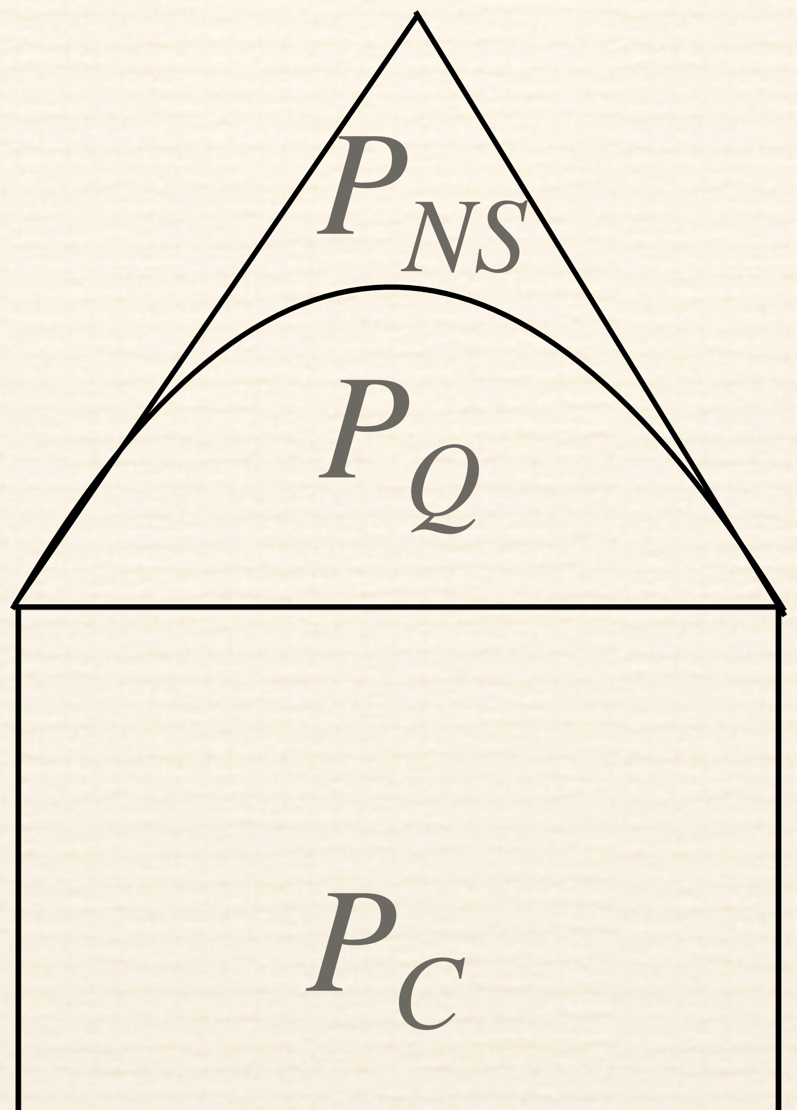
$$P_C \subsetneq P_Q \subseteq P_{NS}$$

$$P_Q \subsetneq P_{NS}?$$

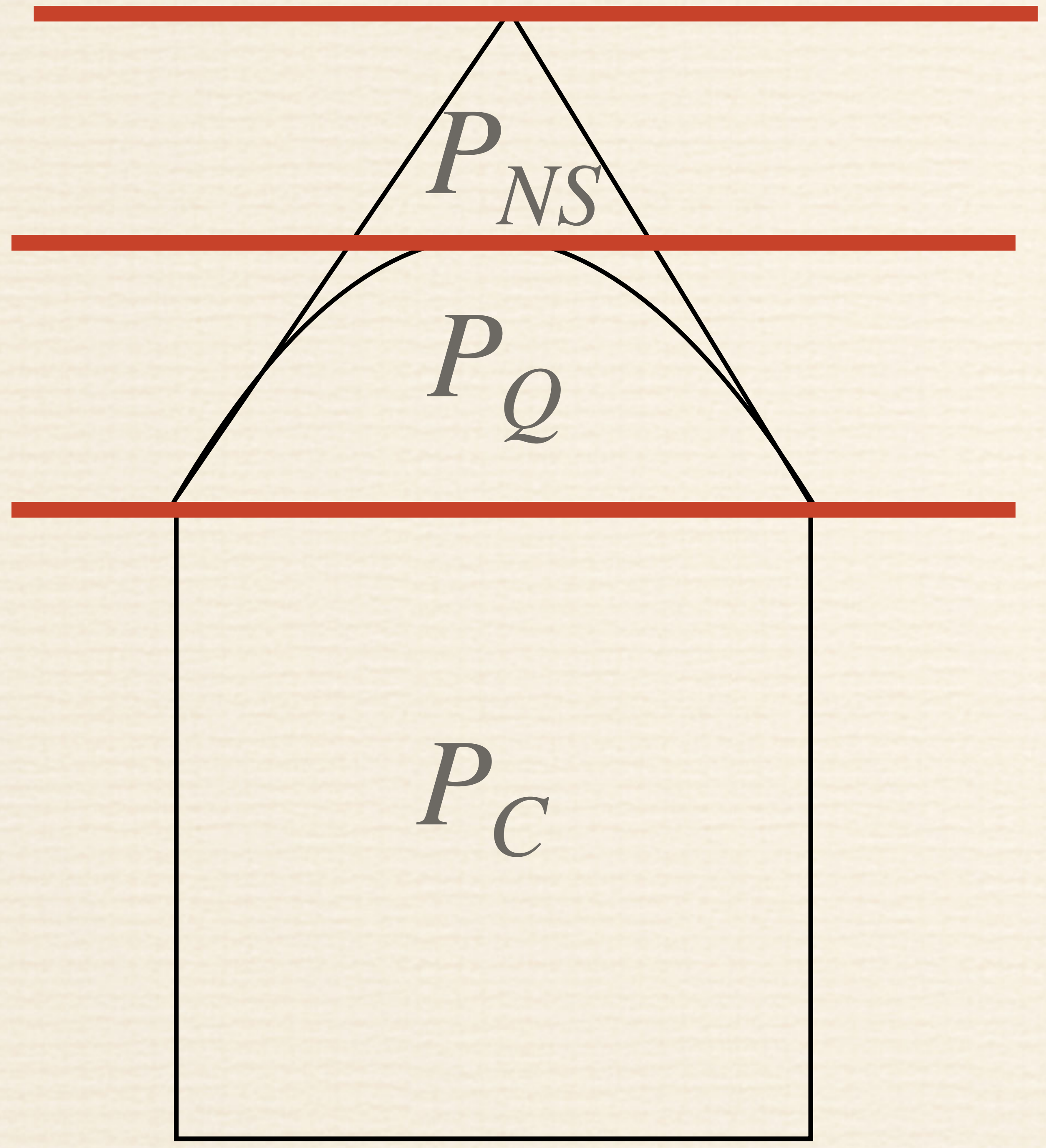
Does the no-signalling principle fully determine the set of quantum correlations?

$$P_Q \subsetneq P_{NS}$$

Cannot derive the set of quantum correlations from no-signalling principle alone



Does the no-signalling principle fully determine the set of quantum correlations?



$$\langle A_0B_0\rangle + \langle A_0B_1\rangle + \langle A_1B_0\rangle - \langle A_1B_1\rangle \leq 4$$

$$\langle A_0B_0\rangle + \langle A_0B_1\rangle + \langle A_1B_0\rangle - \langle A_1B_1\rangle \leq 2\sqrt{2}$$

$$\langle A_0B_0\rangle + \langle A_0B_1\rangle + \langle A_1B_0\rangle - \langle A_1B_1\rangle \leq 2$$

Can use inequalities as witnesses for membership in different sets

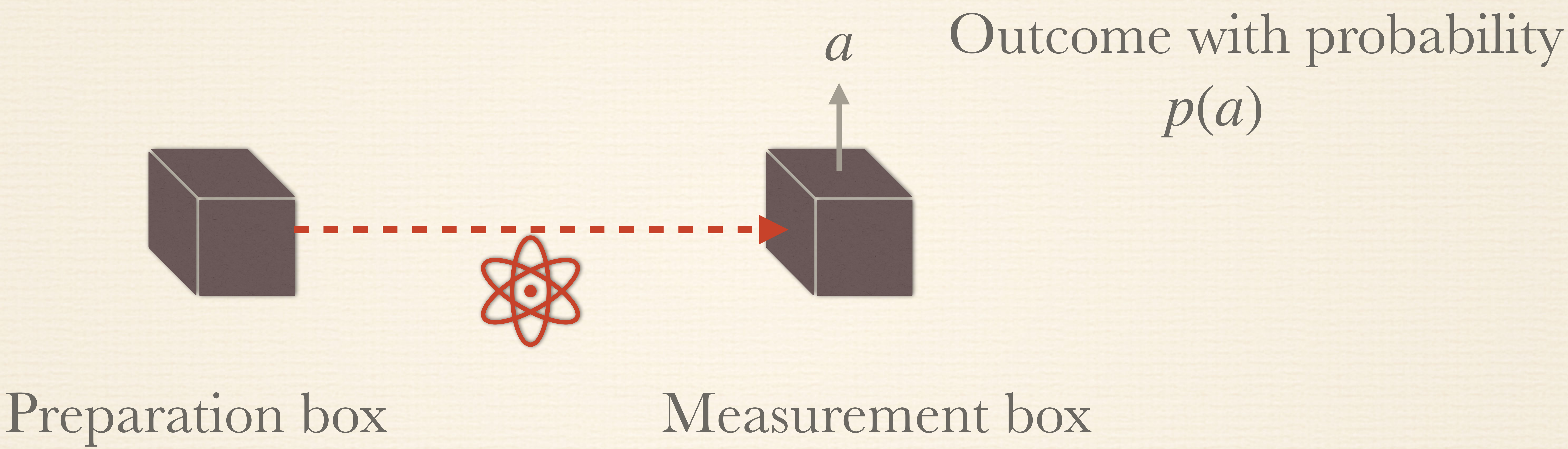
The general lessons

- A physical principle imposes constraints on the possible correlations
- The quantumly realisable correlations are a subset of the general correlations
- Inequalities can be used to witness membership in different sets
- Space-time structure does not single out quantum theory (in this case)

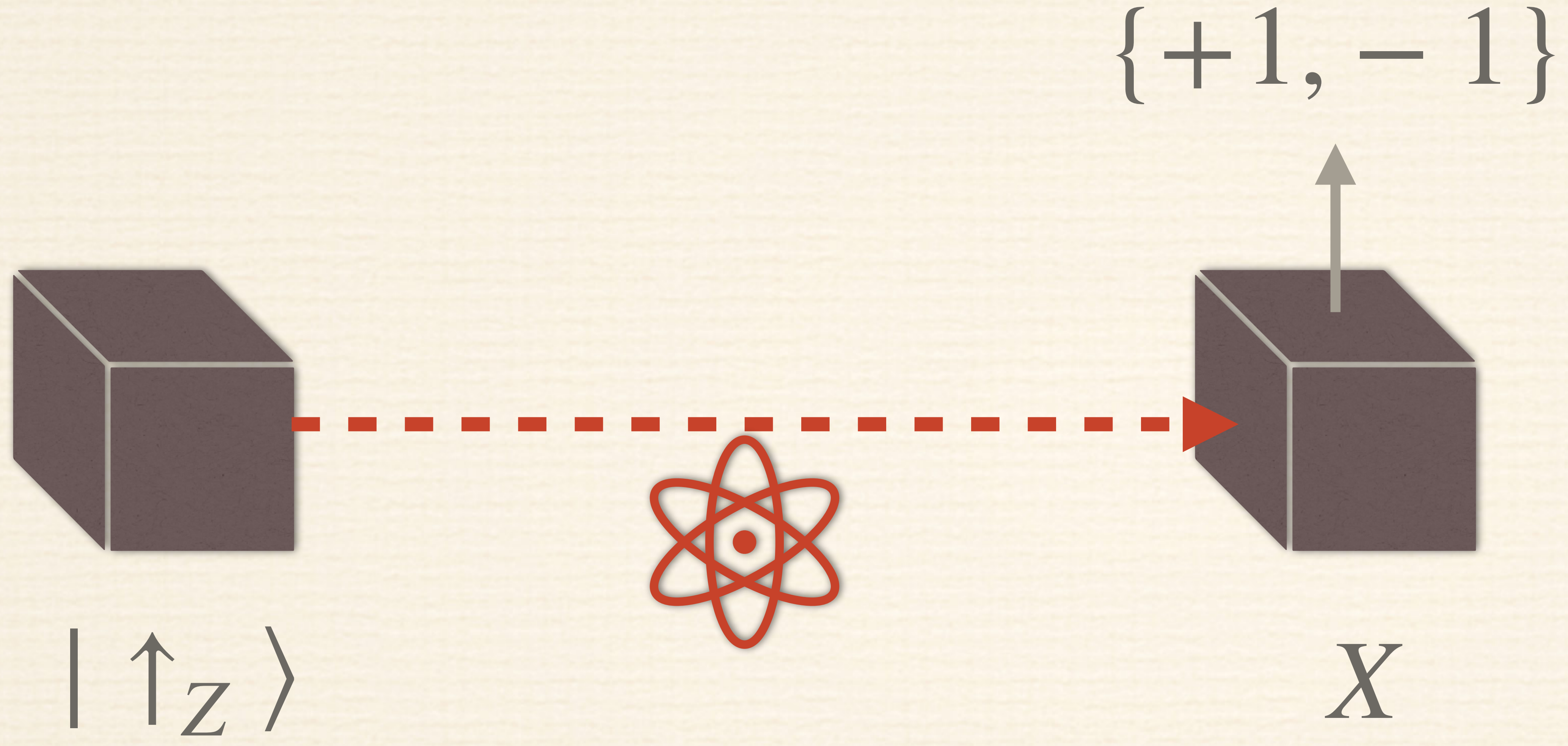
No-signalling is not the only relevant spatio-temporal feature!

- Spatio-temporal symmetry groups are a defining feature of space-time.
- Do space-time symmetries fully constrain probabilistic theories to be quantum or is there a gap?

Prepare and measure scenario

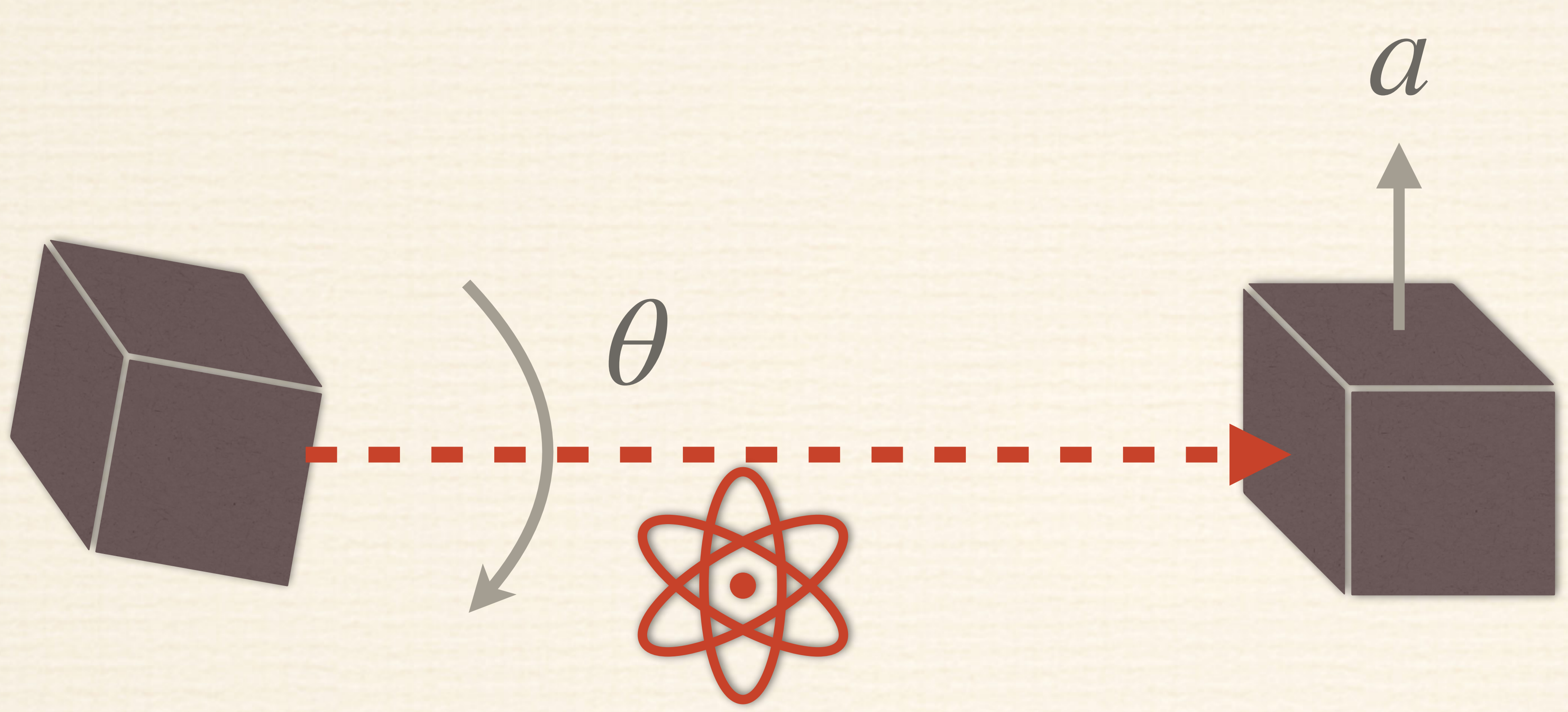


Prepare and measure scenario



$$p(+1 \,|\, X) = p(-1 \,|\, X) = \frac{1}{2}$$

Rotation boxes



Outcome with probability

$p(a|\theta)$

$a \in \{0,1,\ldots,m\}$

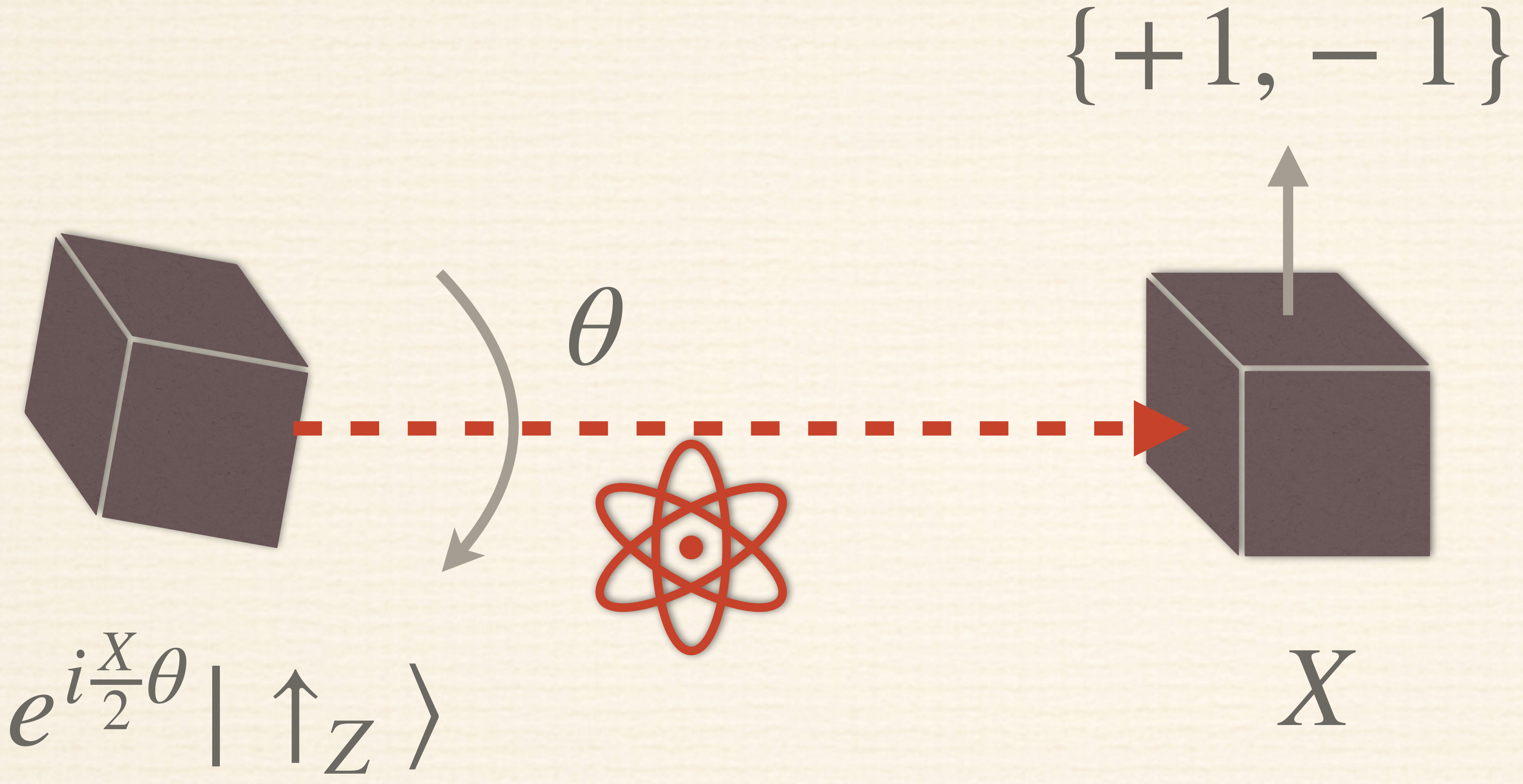
Preparation box

Measurement box

Preparation box has input $\theta \in [0,2\pi)$

Typically input is $x \in \{0,1,\ldots,n\}$

Rotation boxe: example



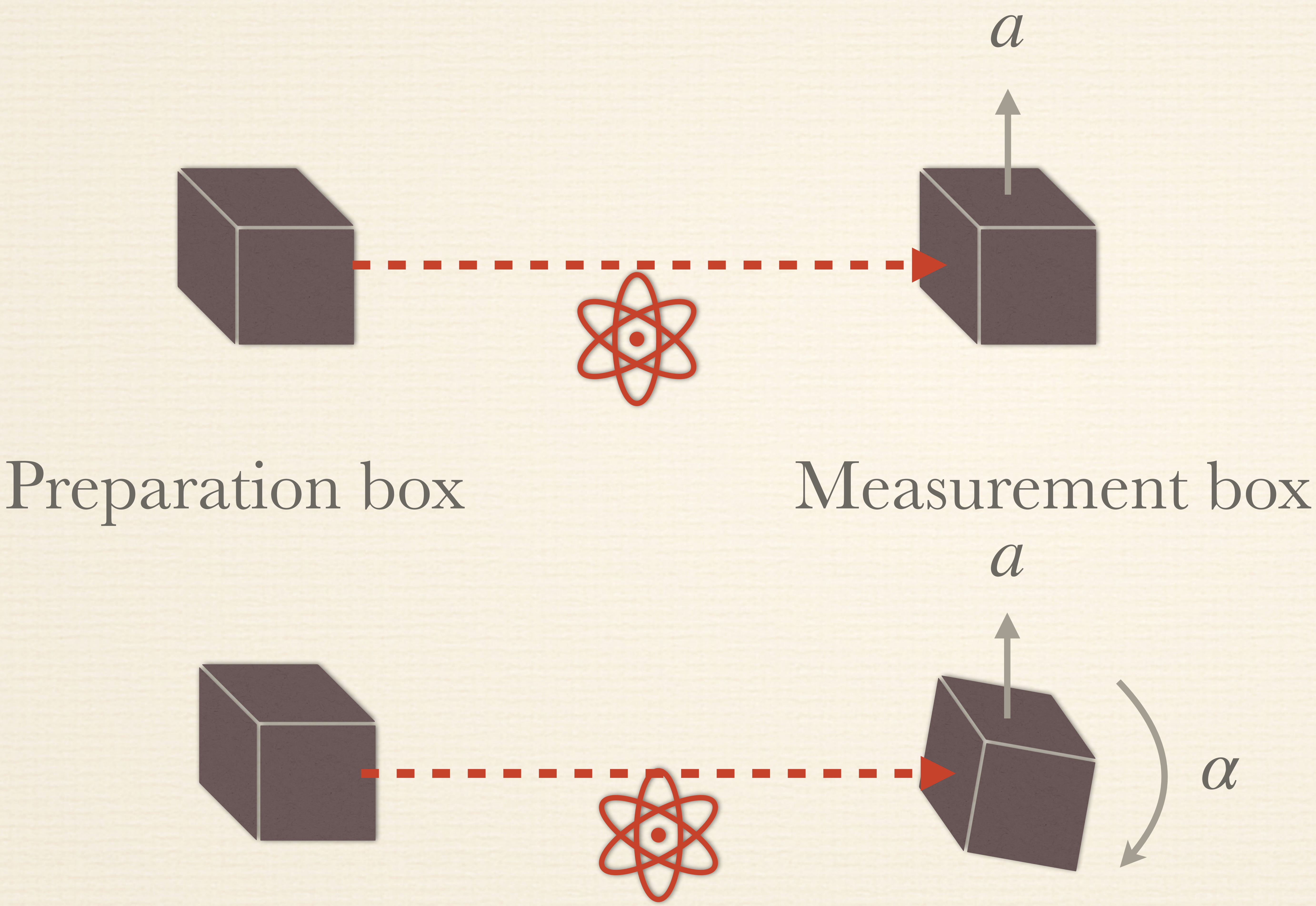
$$p(+1\,|\,\theta) = \frac{1}{2}(\cos(\theta) + 1)$$

Rotation boxes

- A preparation device with input $\theta \in [0,2\pi)$ and a measurement device with outcomes $a \in \{1,...,m\}$ generate a probability distribution $p(a|\theta) \in [0,1]$

- Preparation device and measurement device are initially uncorrelated (e.g. do not share an entangled state): semi-device independent regime.

Prepare and measure scenario

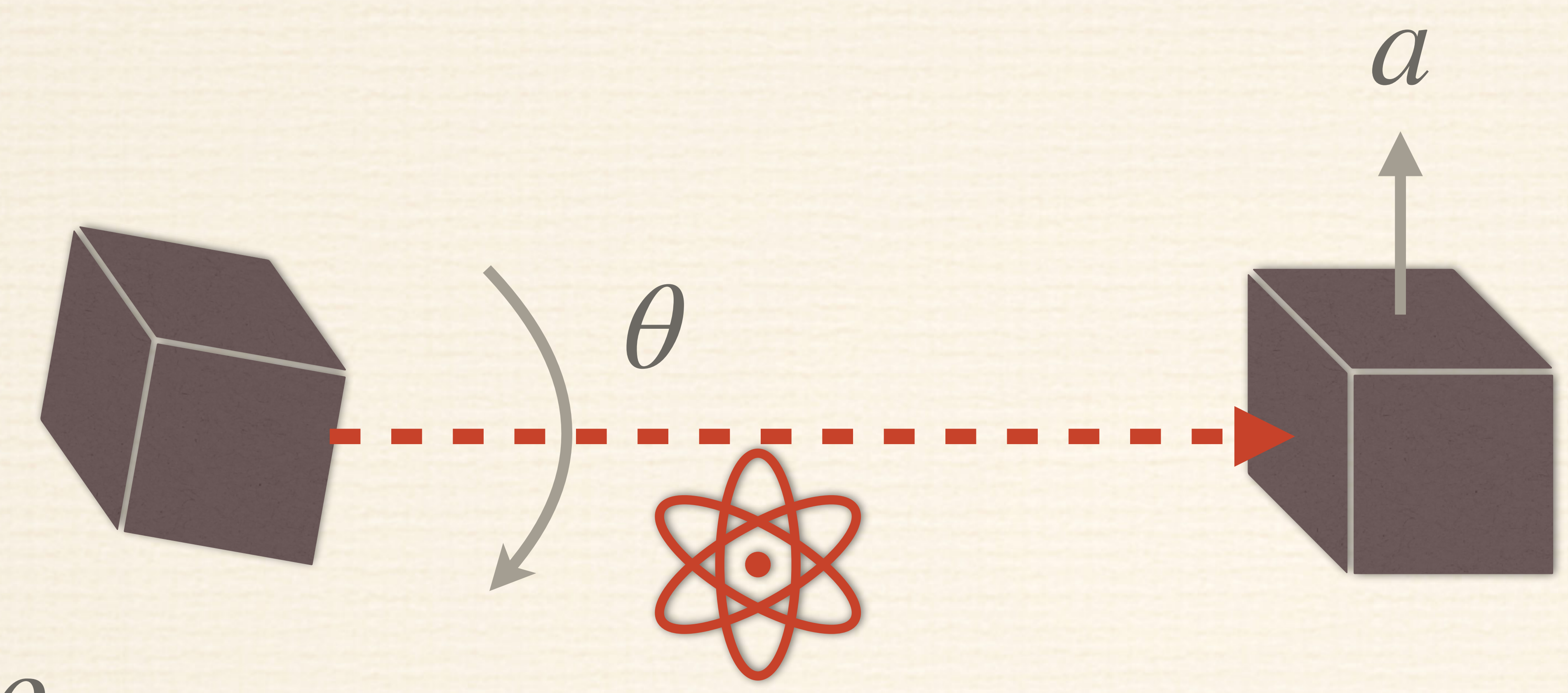


$p(a|\theta)$

$p(a|\theta-\alpha) = q(a|\theta)$

Quantum rotation boxes

System $\mathbb{C}^d$



$$p(a\,|\,\theta) = \mathrm{Tr}(U_\theta \rho U_\theta^\dagger E_a)$$

ρ

Density operator

$\rho \mapsto U_\theta \rho U_\theta^\dagger$

(Projective) unitary representation

$\{E_a\}_{a\in\mathscr{A}}$

POVM

Quantum spin 1/2 correlations (A = 2)

$$U_\theta = e^{iZ\theta} = \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}$$

$$p(a\,|\,\theta) = \langle\psi\,|\,U_\theta E_a U_\theta^\dagger\,|\,\psi\rangle = c_0 + c_1\cos(\theta) + s_1\sin(\theta)$$

$$\mathcal{Q}^2_{\frac{1}{2}} = \{p(a\,|\,\theta) = \{\langle\psi\,|\,U_\theta E_a U_\theta^\dagger\,|\,\psi\rangle\,|\,\rho, E_a\}$$

Set of spin 1/2 quantum correlations

General spin 1/2 correlations

$$\mathscr{R}^2_{\frac{1}{2}} = \{p(a|\theta) = c_0 + c_1\cos(\theta) + s_1\sin(\theta) \,|\, c_0, c_1, s_1 \in \mathbb{R}, p(a|\theta) \in [0,1]\}$$

$$\mathscr{Q}^2_{\frac{1}{2}} \subseteq \mathscr{R}^2_{\frac{1}{2}}$$

$$\mathscr{Q}^2_{\frac{1}{2}} = \mathscr{R}^2_{\frac{1}{2}}?$$

Can every trigonometric polynomial of order 1 which is a valid probability be generated by a spin 1/2 quantum particle?

$\mathcal{Q}^2_{\frac{1}{2}} = \mathcal{R}^2_{\frac{1}{2}}$?

Yes!

Every $p(a\,|\,\theta) = c_0 + c_1\cos(\theta) + s_1\sin(\theta) \in [0,1]$ can be realised on a qubit

Spin J quantum correlations

General projective unitary representation of SO(2):

$$U_\theta = \bigoplus_{j=-J}^{+J} \mathbb{I}_{N_j} e^{ij\theta}$$

$$J \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$$

Observe only integers or half integers can occur in the sum.

Spin J quantum correlations

$$U_\theta = e^{i \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}}$$

$$U_\theta = e^{i \begin{pmatrix} -\frac{5}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{2} \end{pmatrix}}$$

Spin J quantum correlations

$$p(a|\theta) = \mathrm{Tr}(U_\theta \rho U_\theta^\dagger E_a)$$

$$U_\theta = \bigoplus_{j=-J}^{+J} \mathbb{I}_{N_j} e^{ij\theta}$$

$$\mathcal{Q}_J = \{p(a|\theta) \,|\, p(a|\theta) = \mathrm{Tr}(U_\theta \rho U_\theta^\dagger E_a)\}$$

$$p(a|\theta) \in \mathcal{Q}_J \implies p(a|\theta) = c_0 + \sum_{j=1}^{2J} c_j \cos(j\theta) + s_j \sin(j\theta)$$

Theorem

$$p(a\,|\,\theta) \in \mathbb{Q}_J \implies \exists\,|\psi\rangle \in \mathbb{C}^{2J+1}$$

$$p(a\,|\,\theta) = \langle\psi\,|\,U_\theta^\dagger E_a U_\theta\,|\,\psi\rangle$$

where

$$U_\theta = e^{i\theta Z},\ Z = \mathrm{diag}(-J, -J+1, \ldots, J-1, J)$$

J integer or half-integer

Theorem

$$U_\theta = e^{i\begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}} \mapsto e^{i\begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}}$$

Spin J quantum correlations

$$p(a\,|\,\theta) = \langle\psi|\,U_\theta^\dagger E_a U_\theta\,|\psi\rangle$$

where

$$U_\theta = e^{i\theta Z},\ Z = \mathrm{diag}(-J, -J+1, \ldots, J-1, J),\ |\psi\rangle \in \mathbb{C}^{2J+1}$$

$$p(a\,|\,\theta) = c_0 + \sum_{j=1}^{2J} c_j \cos(j\theta) + s_j \sin(j\theta)$$

$$\mathcal{Q}_J \subset \mathcal{Q}_{J+\frac{1}{2}}$$

General spin J correlations

$$\mathscr{R}_J := \{p(a|\theta) = c_0 + \sum_{j=1}^{2J} c_j \cos(j\theta) + s_j \sin(j\theta) \,|\, p(a|\theta) \in [0,1]\}$$

$$\mathscr{Q}_J \subseteq \mathscr{R}_J$$

Main question of the work

When does $\mathcal{Q}_J = \mathcal{R}_J$?

Does the requirement of rotational covariance and fixed spin constrain probabilities to be quantum mechanical?

Analogous to question of whether the no-signalling constraint implies quantum probabilities

Spin 1 quantum correlations

$$U_\theta = \begin{pmatrix} e^{-i\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} \qquad |\psi\rangle \in \mathbb{C}^3$$

$$p(a\,|\,\theta) = \langle\psi\,|\,U_\theta^\dagger E_a U_\theta\,|\,\psi\rangle$$

$$p(a\,|\,\theta) = c_0 + c_1\cos(\theta) + s_1\sin(\theta) + c_2\cos(2\theta) + s_2\sin(2\theta)$$

Trigonometric polynomial of degree 2

The case J = 1

$$p(a\,|\,\theta) = c_0 + c_1\cos(\theta) + s_1\sin(\theta) + c_2\cos(2\theta) + s_2\sin(2\theta)$$

$$p(a\,|\,\theta) = \begin{pmatrix} c_0 & c_1 & s_1 & c_2 & s_2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \cos(\theta) \\ \sin(\theta) \\ \cos(2\theta) \\ \sin(2\theta) \end{pmatrix}$$

State space

$$\Omega_1 := \mathrm{Conv}\left(\left\{ \begin{pmatrix} 1 \\ \cos(\theta) \\ \sin(\theta) \\ \cos(2\theta) \\ \sin(2\theta) \end{pmatrix} \middle| \theta \in [0,2\pi) \right\} \right)$$

Orbitope: convex hull of the orbit of a group acting on a vector

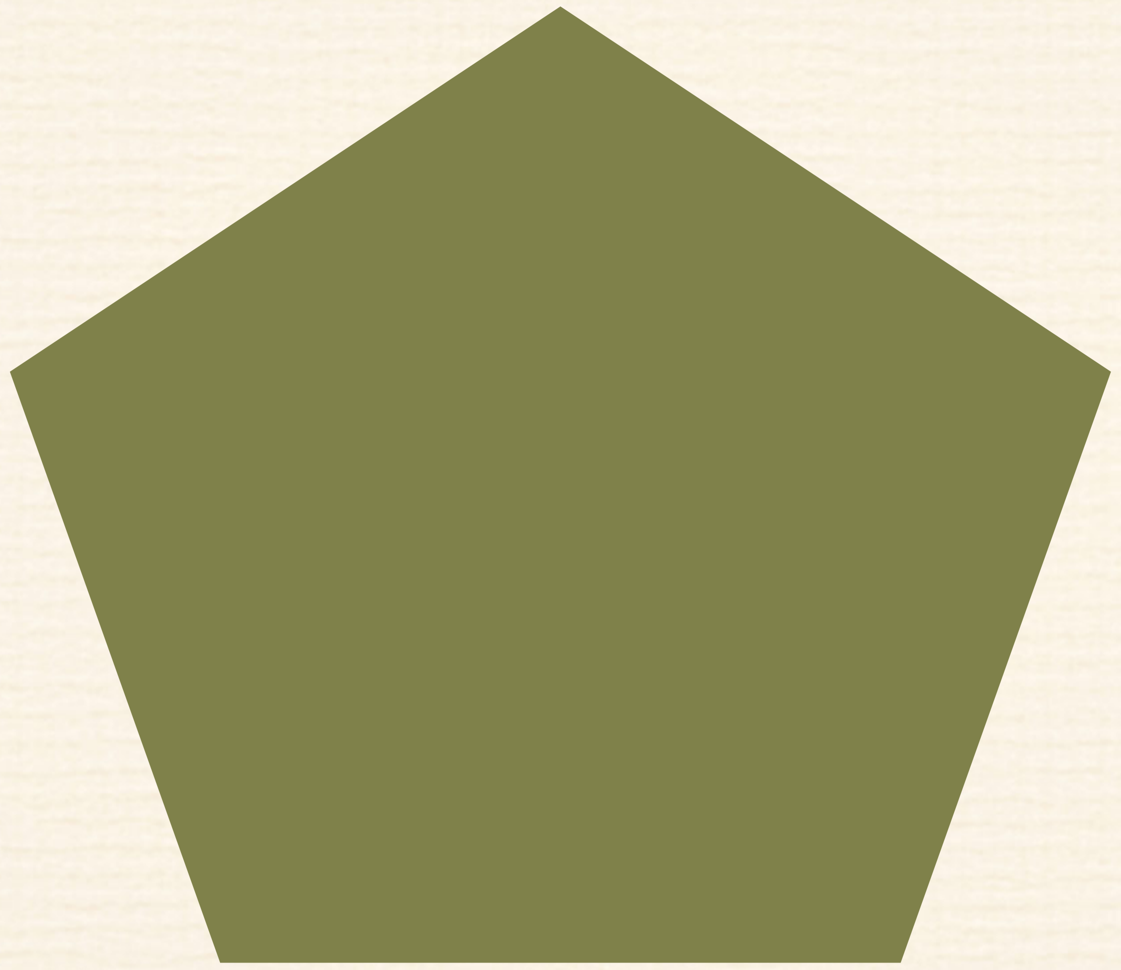
Effect space

$$\mathscr{E}_1 := \{(c_0 \quad c_1 \quad s_1 \quad c_2 \quad s_2) \,|\, (c_0 \quad c_1 \quad s_1 \quad c_2 \quad s_2) \cdot \begin{pmatrix} 1 \\ \cos(\theta) \\ \sin(\theta) \\ \cos(2\theta) \\ \sin(2\theta) \end{pmatrix} \in [0,1], \ \forall \theta \in [0,2\pi)\}$$

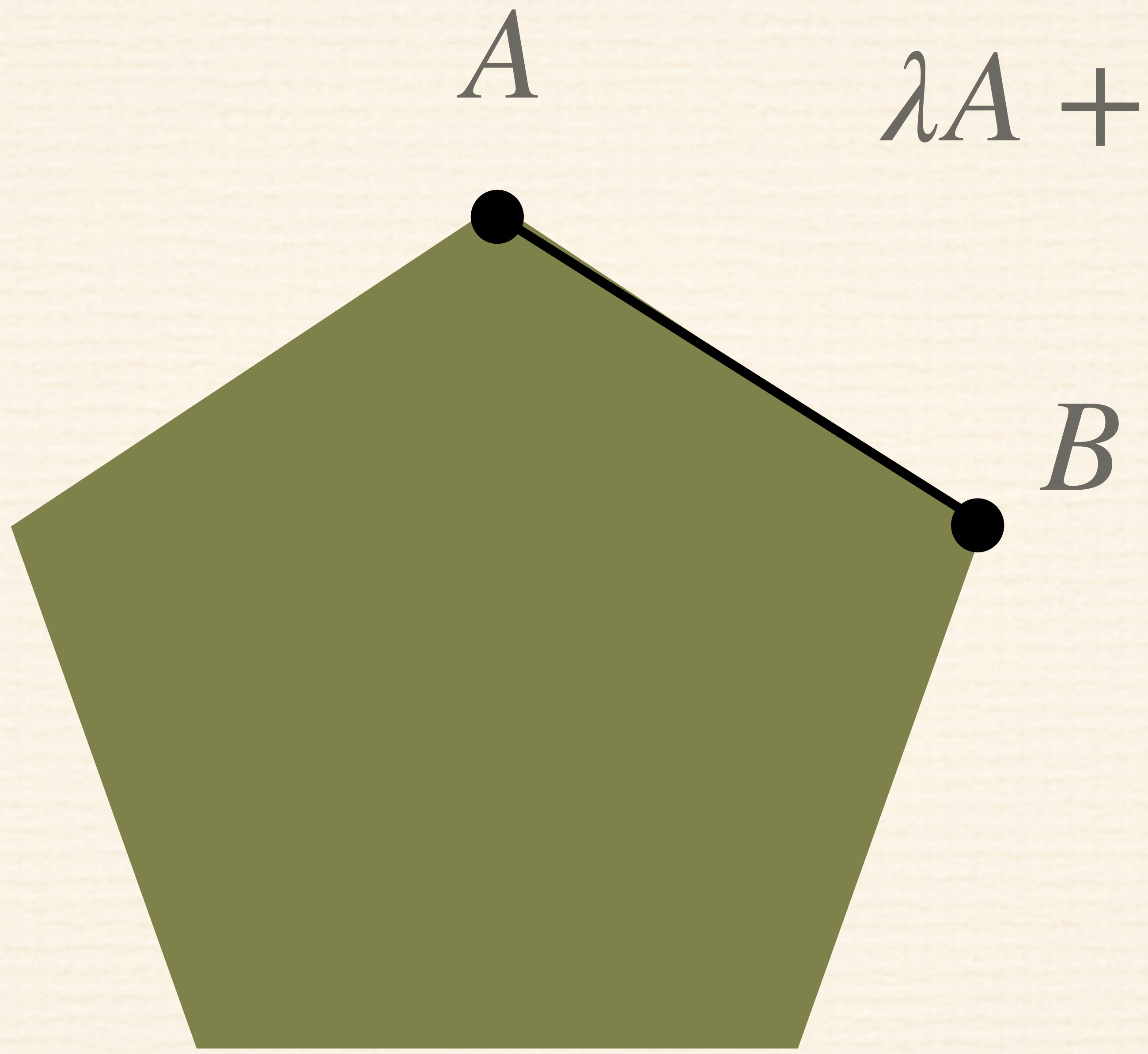
$$\mathscr{R}_1 := \{(c_0, c_1, s_1, c_2, s_2) \,|\, c_0 + c_1 \cos(\theta) + s_1 \sin(\theta) + c_2 \cos(2\theta) + s_2 \sin(2\theta) \in [0,1]\}$$

$$\mathscr{R}_1 \simeq \mathscr{E}_1$$

Convex sets

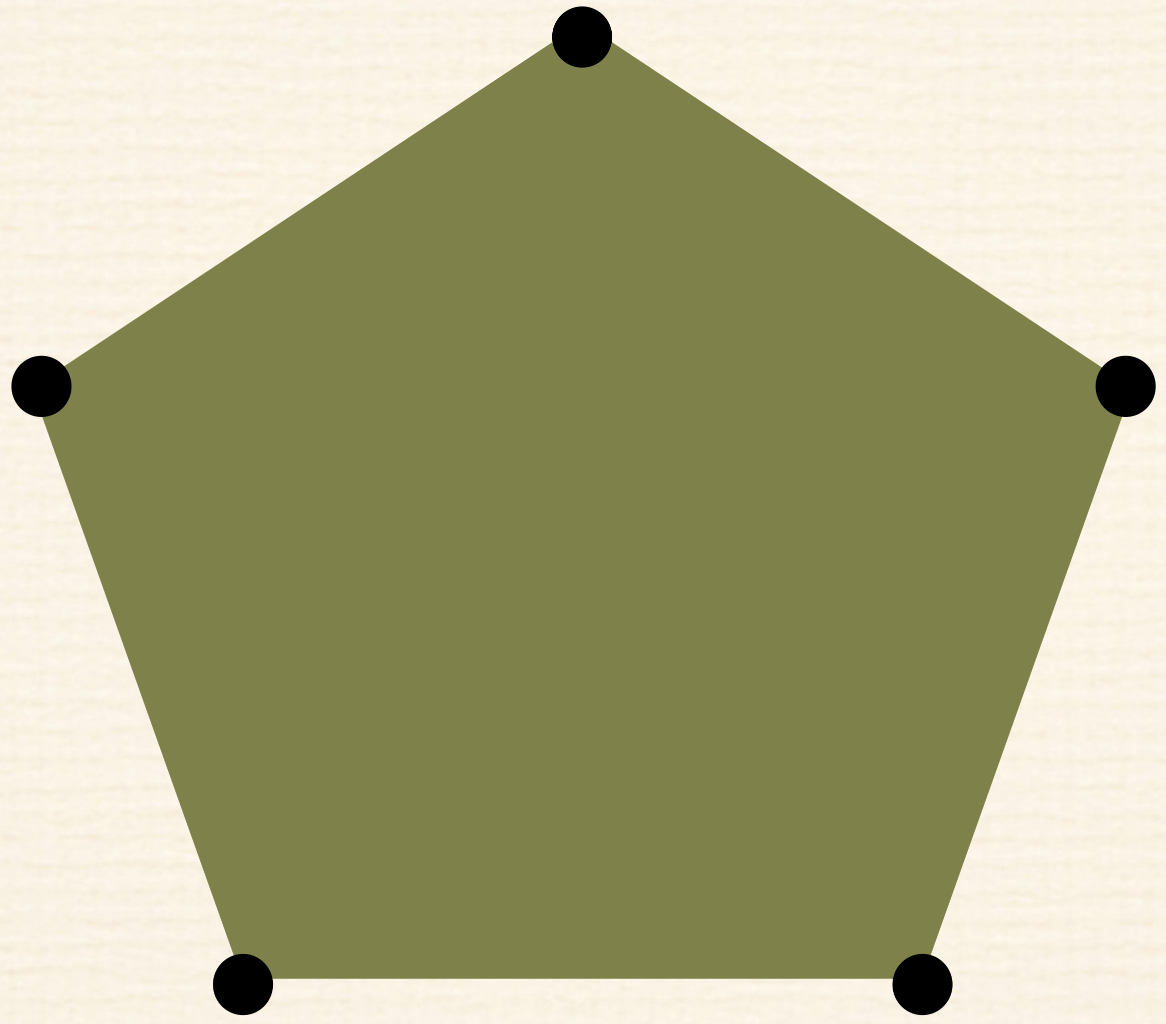


Convex sets

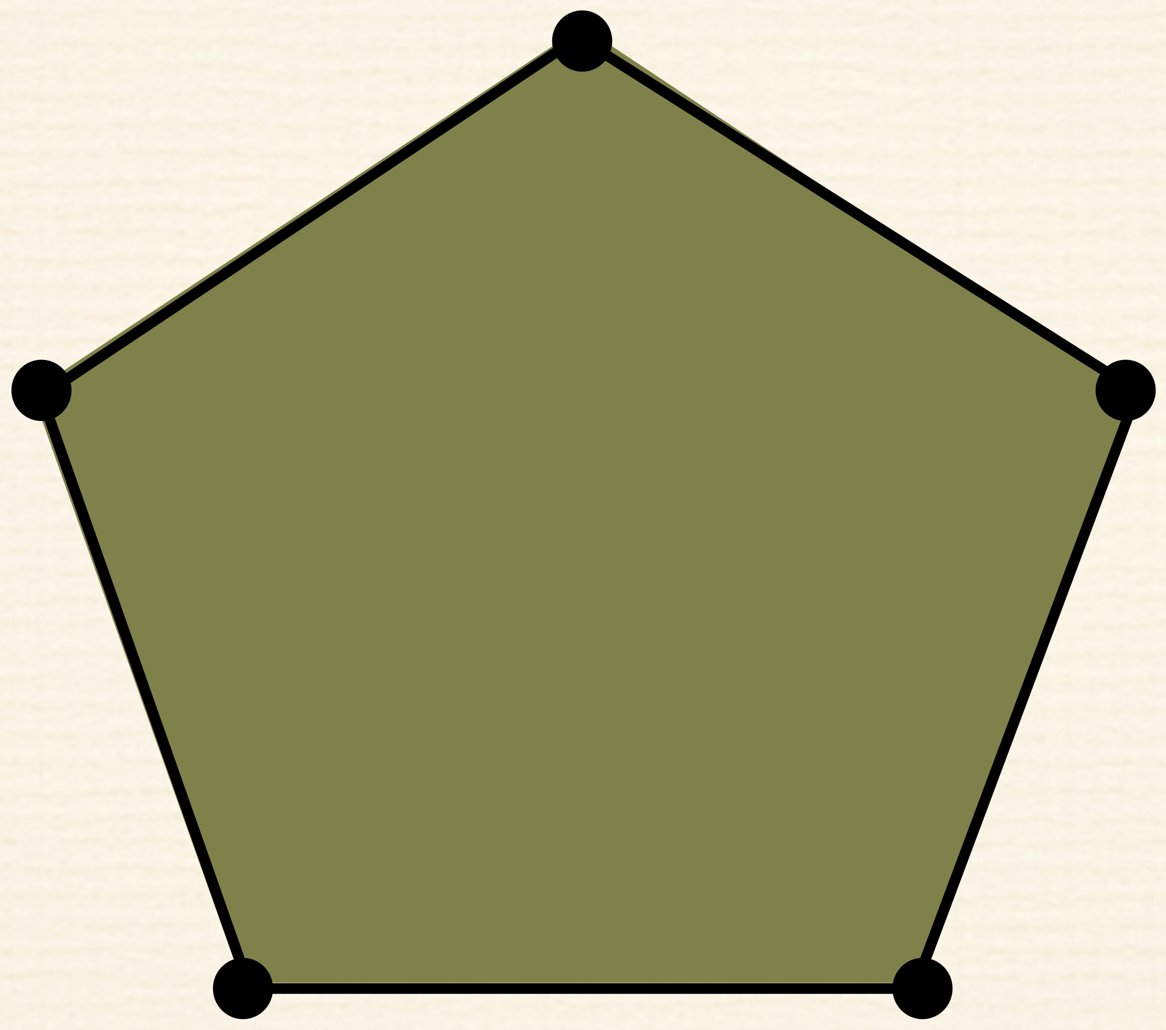


$\lambda A + (1 - \lambda)B, \ \lambda \in [0,1]$

Convex sets: extremal points



Convex sets: faces



Carathéodory orbitope

$$\Omega_1 := \mathrm{Conv}\left(\left\{ \begin{pmatrix} 1 \\ \cos(\theta) \\ \sin(\theta) \\ \cos(2\theta) \\ \sin(2\theta) \end{pmatrix} \middle| \theta \in [0,2\pi) \right\} \right)$$

Well characterised as a convex set (facial structure known)

Space of J =1 correlations

Want to characterise the dual space

$$\mathscr{R}_1 := \{(c_0, c_1, s_1, c_2, s_2) \,|\, c_0 + c_1 \cos(\theta) + s_1 \sin(\theta) + c_2 \cos(2\theta) + s_2 \sin(2\theta) \in [0,1]\}$$

Space of J =1 correlations

Characterise all extremal points of $\mathscr{R}_1$

Prove that all extremal points of $\mathscr{R}_1$ are quantumly realisable with a spin 1 system

$$\mathscr{R}_1 = \mathscr{Q}_1$$

The case J = 3/2

$$p(a\,|\,\theta) = c_0 + c_1\cos(\theta) + s_1\sin(\theta) + c_2\cos(2\theta) + s_2\sin(2\theta) + c_3\cos(3\theta) + s_3\sin(3\theta)$$

Theorem

$$p(a\,|\,\theta) \in \mathcal{Q}_{\frac{3}{2}} \implies c_2 + s_3 \leq \frac{1}{\sqrt{3}} \lesssim 0.5774.$$

Uses semi-definite program and analogue to Almost Quantum correlations

The case J = 3/2

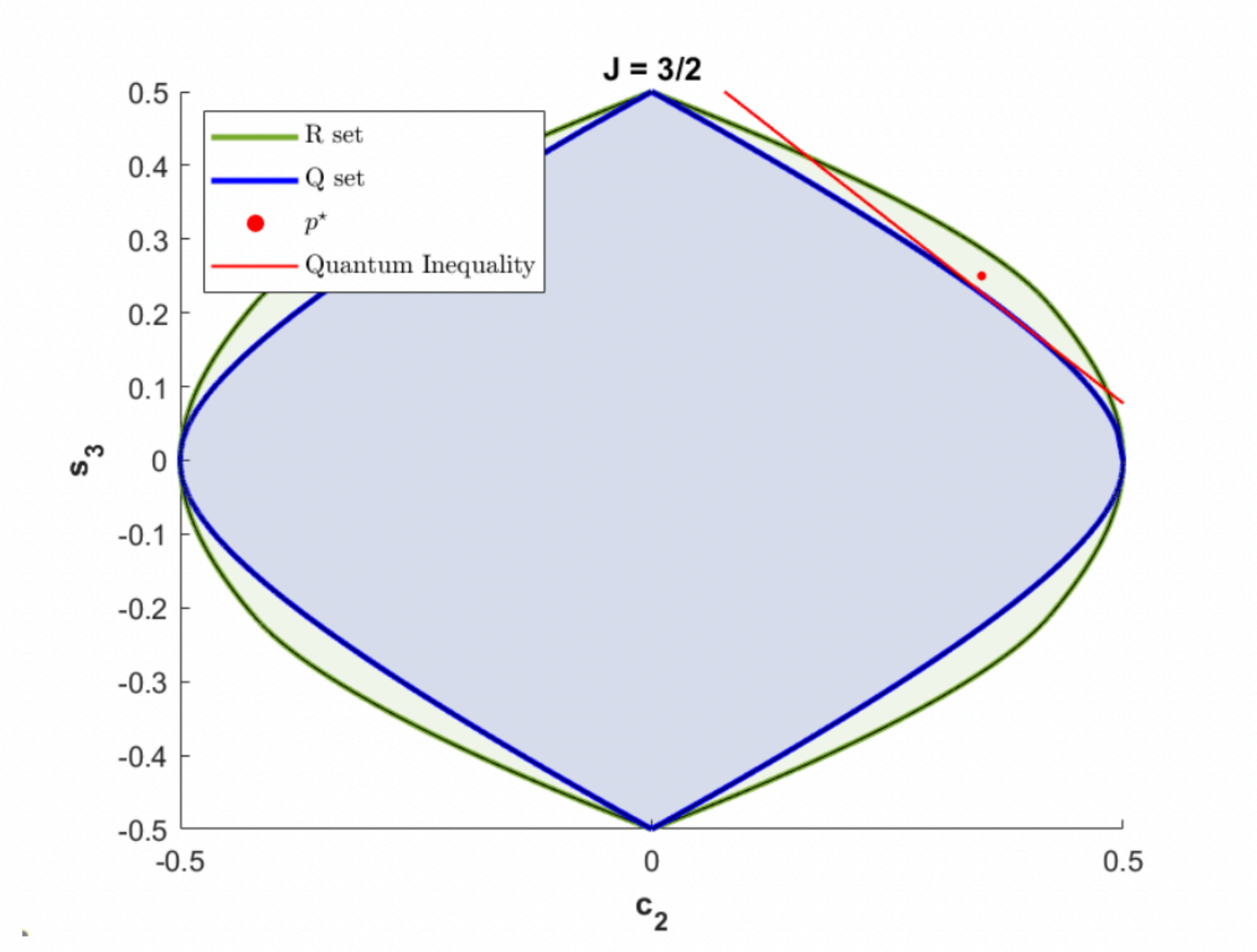
$$p^\star(\theta) := \frac{2}{5} + \frac{1}{4}\sin\theta + \frac{7}{20}\cos(2\theta) + \frac{1}{4}\sin(3\theta) \in [0,1]$$

$$p^\star(\theta) \in \mathscr{R}_{\frac{3}{2}}$$

$$c_2 + s_3 = 0.6$$

$$p(a\,|\,\theta) \in \mathcal{Q}_{\frac{3}{2}} \implies c_2 + s_3 \leq \frac{1}{\sqrt{3}} \lesssim 0.5774.$$

$$p^\star(\theta) \notin \mathcal{Q}_{\frac{3}{2}}$$



$$p(a\,|\,\theta) = c_0 + c_1\cos(\theta) + s_1\sin(\theta) + c_2\cos(2\theta) + s_2\sin(2\theta) + c_3\cos(3\theta) + s_3\sin(3\theta)$$

The case $J \geq 2$

$$P \in \mathcal{Q}_J \quad \Longrightarrow \quad (c_{2J-1} + s_{2J})[P] \leq \beta = \frac{1}{\sqrt{3}}$$

$$P_J^\star(\theta) := \sum_{k=-2J}^{2J} a_k e^{ik\theta}, \; a_{-k} = \overline{a_k} \;\; a_0 = \frac{1}{2} \;\; a_{2J} = -\frac{i}{8}$$

$s_{2J} + c_{2J-1} = 5/8 > \beta$

$$a_{2J-1-2m} = \frac{3}{16}\left(-\frac{1}{4}\right)^m \qquad m = 0, \ldots, \lfloor J-1 \rfloor,$$

$$a_{2J-2-2l} = -\frac{3i}{32}\left(-\frac{1}{4}\right)^l \qquad l = 0, \ldots, \lceil J-2 \rceil .$$

Summary

$$\mathscr{R}_0 = \mathscr{Q}_0$$

$$\mathscr{R}_{\frac{1}{2}} = \mathscr{Q}_{\frac{1}{2}}$$

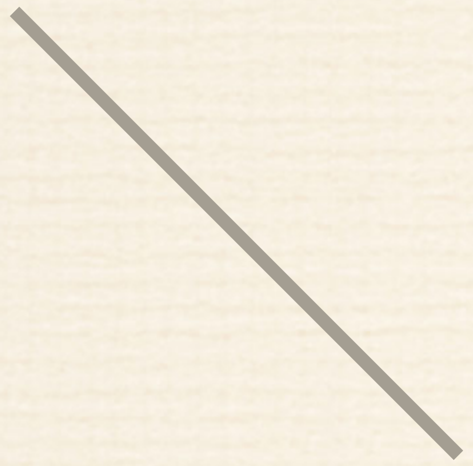
$$\mathscr{R}_1 = \mathscr{Q}_1$$

$$\mathscr{R}_J \subsetneq \mathscr{Q}_J, J \geq \frac{3}{2}$$

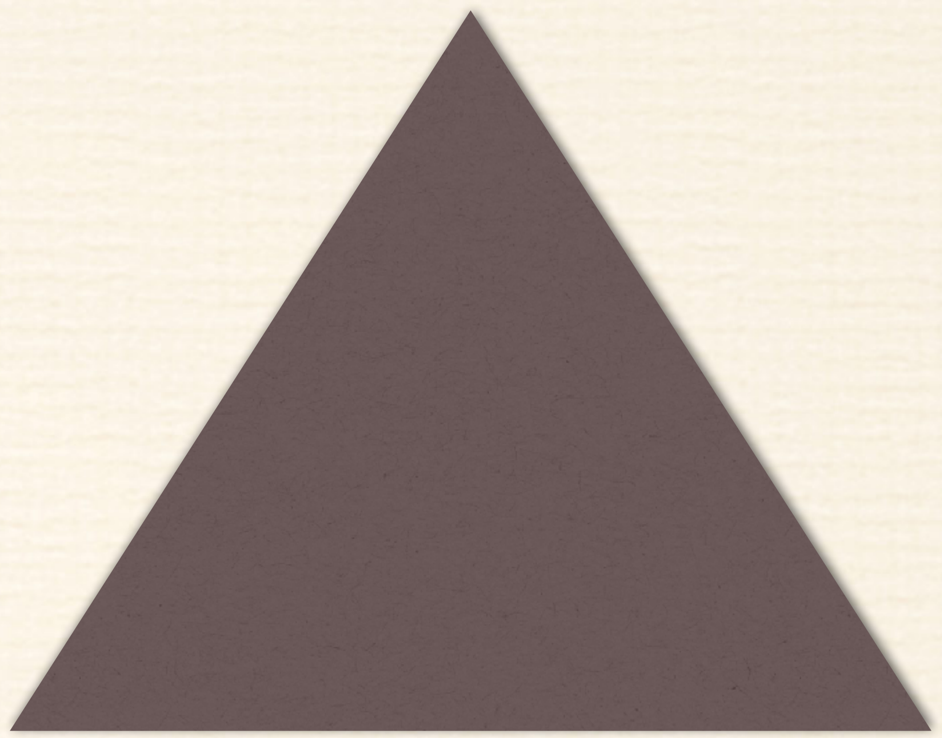
How important is the fixed J assumption

- Very!
- $\mathscr{R}_J$ can be realised by an infinite spin quantum system $L^2(S_1)$
- Can also be realised by a classical system with configuration space S_1
- No finite dimensional classical systems have a non-trivial representation of SO(2)

Finite dimensional classical systems

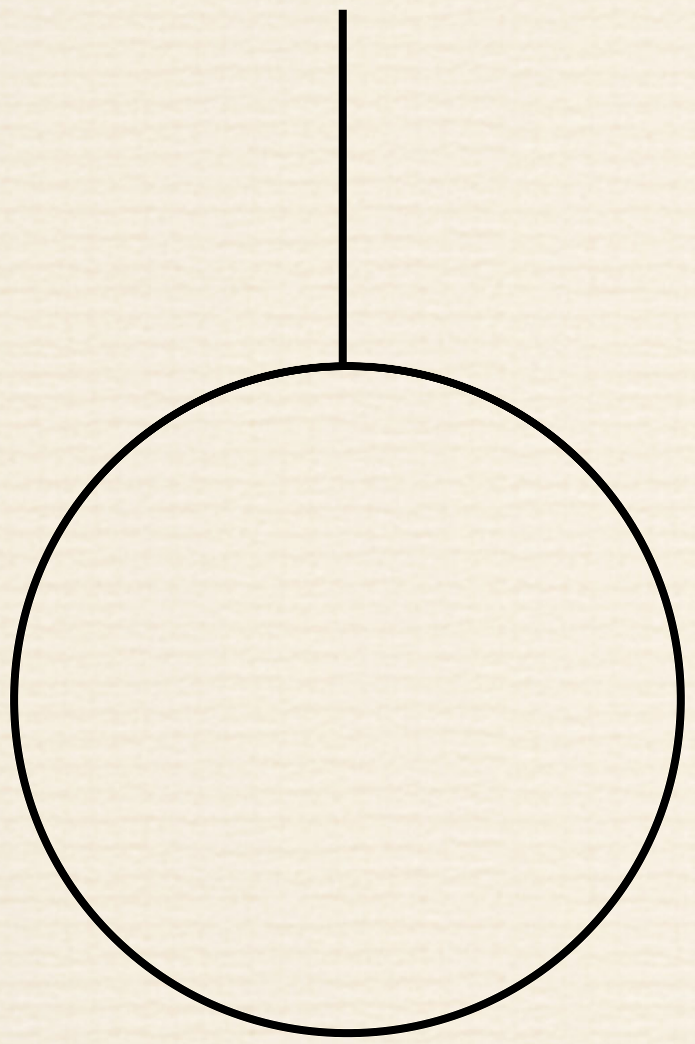


$$\begin{pmatrix} p(0) \\ p(1) \end{pmatrix}$$



$$\begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix}$$

Infinite dimensional classical system



States are probability measures on circle.

Effects are response functions: $e : S^2 \to [0,1]$

$\delta_\theta \mapsto \delta_{\theta+\theta'}$

$$p(a|\theta) = \int p(a|\theta')\delta_\theta$$

Semi-device independence

- For single systems cannot have full device independence.
- Any statistics can be simulated by a large enough classical system.
- Impose some constraint: e.g Hilbert space dimension. In this case a notion of generalised spin.

Conclusion

- SO(2) rotations do not constrain correlations to be quantum for $J \geq \frac{3}{2}$
- SO(2) covariance + spin 1/2 or 1 implies quantum correlations.
- Future work: extension to SO(3) or Lorentz group.