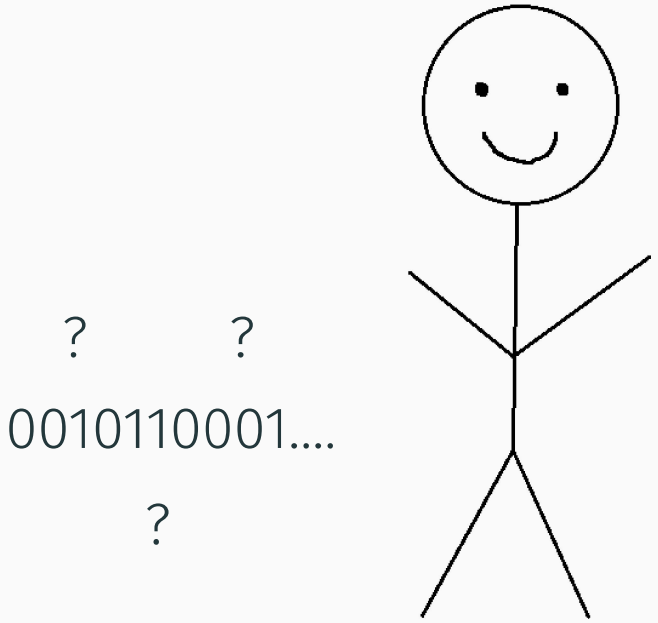
Maximal intrinsic randomness of a quantum state

Young Quantum Information Scientists

Fionnuala Curran

8th November 2024

ICFO–The Institute of Photonic Sciences



Key questions

- What is **intrinsic** about quantum randomness?
- How can we quantify intrinsic randomness?
- How do we extract **maximal intrinsic randomness** from a given quantum system?
- Why do we care?

Based on…

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Letter

Maximal intrinsic randomness of a quantum state

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One of the most counterintuitive aspects of quantum theory is its claim that there is "intrinsic" randomness in the physical world. Quantum information science has greatly progressed in the study of intrinsic, or secret, quantum randomness in the past decade. With much emphasis on device-independent and semi-device-independent bounds, one of the most basic questions has escaped attention: how much intrinsic randomness can be extracted from a given state ρ, and what measurements achieve this bound? We answer this question for three different randomness quantifiers: the conditional min-entropy, the conditional von Neumann entropy, and the conditional max-entropy. For the first, we solve the min-max problem of finding the projective measurement that minimizes the maximal guessing probability of an eavesdropper. The result is that one can guarantee an amount of conditional min-entropy $H^*_{\min} = -\log_2 P^*_{\text{guess}}(\rho)$ with $P^*_{\text{guess}}(\rho) = \frac{1}{2}(\text{tr}\sqrt{\rho})^2$ by performing suitable projective measurements. For the conditional von Neumann entropy, we find that the maximal value is $H^* = \log_2 d - S(\rho)$, with $S(\rho)$ the von Neumann entropy of ρ, while for the conditional max-entropy, we find the maximal value $H^*_{\max} = \log_2 d + \log_2 \lambda_{\max}(\rho)$, where $\lambda_{\max}(\rho)$ is the largest eigenvalue of ρ. Optimal values for $H^*_{\min}$, H^* and $H^*_{\max}$ are achieved by measuring in any basis that is unbiased with respect to the eigenbasis of ρ, as well as by other, less intuitive, measurements.

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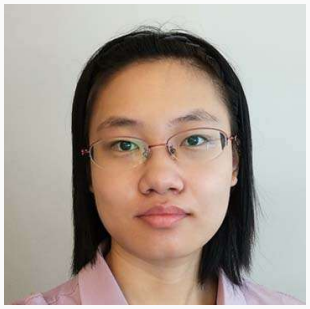
Introduction. One of the core differences between classical and quantum physics is the latter's probabilistic character, which is irreducible to ignorance of underlying variables. This difference has fundamental implications for our worldview, but it is also attractive as a natural source of randomness for practical uses. Indeed, Geiger counting was already used as a source of physical randomness in the second half of the 20th century. In the past two decades, with the development of quantum information science, a large number of quantum random number generators (QRNGs) have been designed, and many have been implemented, usually with light (see [1,2] for comprehensive reviews). The *amount of randomness* is naturally captured by the *guessing probability* P_{guess}: the higher the probability that the random variable is guessed, the smaller the randomness. This intuitive characterization was found to have operational meaning: the *min-entropy* $H_{\min} = -\log_2 P_{\text{guess}}$ quantifies (informally) the fraction of perfect coin tosses that can be extracted from a string generated by the available source. But randomness is not an absolute notion: one has to specify *for whom* the source should be partly unpredictable. For mere sampling purposes, it might be sufficient to take the observed probabilities at face value; for cryptographic applications, however, one needs to estimate the probability that *an adversary, Eve,* guesses the outcomes. The resulting randomness is called *secret randomness*, or *intrinsic randomness*.

The computation of intrinsic randomness using quantum resources and against a quantum adversary has been studied from different perspectives. When considering a user with classical data correlated with quantum information in the hands of an adversary, the min-entropy quantifies the amount of perfect random bits that the user can establish [3]. The question was also addressed for the task of quantum key distribution, which is the extraction of secret *shared* randomness. It was in this context that the idea of device-independent certification was born: the possibility of bounding the amount of randomness in a black-box setting, based on the observation of Bell-nonlocal correlations [4]. Next, it was noticed that device-independent certification can be performed for randomness as well [5,6], providing the first disruptive case for quantum randomness in a non-shared setting [7]. This breakthrough happened as the race to demonstrate loophole-free Bell tests was taking up speed. There followed an explosion of designs and implementations of QRNGs certifiable under various assumptions, from device-independent (disruptive, but hard to implement), to semi-device-independent in various forms, to fully characterized (practical and fast, but requiring a precise modeling



Meng Shuyang

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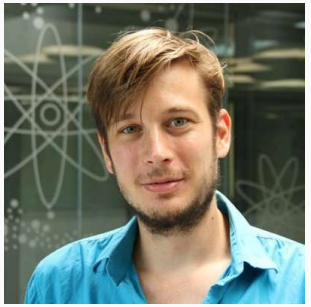

Máté Farkas

Valerio Scarani

Antonio Acín

What is randomness?

randomness

noun [U]

UK /ˈræn.dəm.nəs/ US /ˈræn.dəm.nəs/

Add to word list

the quality of being random (= happening, done, or chosen by chance rather than according to a plan):

- Randomness is important for simulations, algorithms and cryptography.
- Distinct concepts: **statistical** and **intrinsic** randomness.

Statistical randomness

- Are there any **patterns** in the string?

- Tested by *statistical tests* like the Shannon entropy,

$$H(X) = -\sum_{x=0,1} p(x) \log_2 p(x) \,.$$

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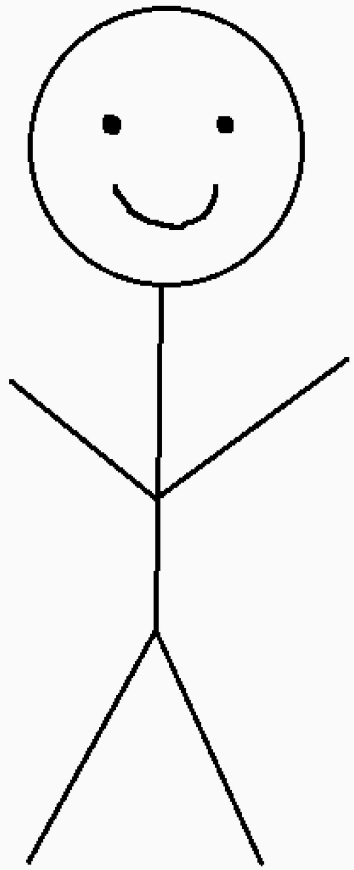
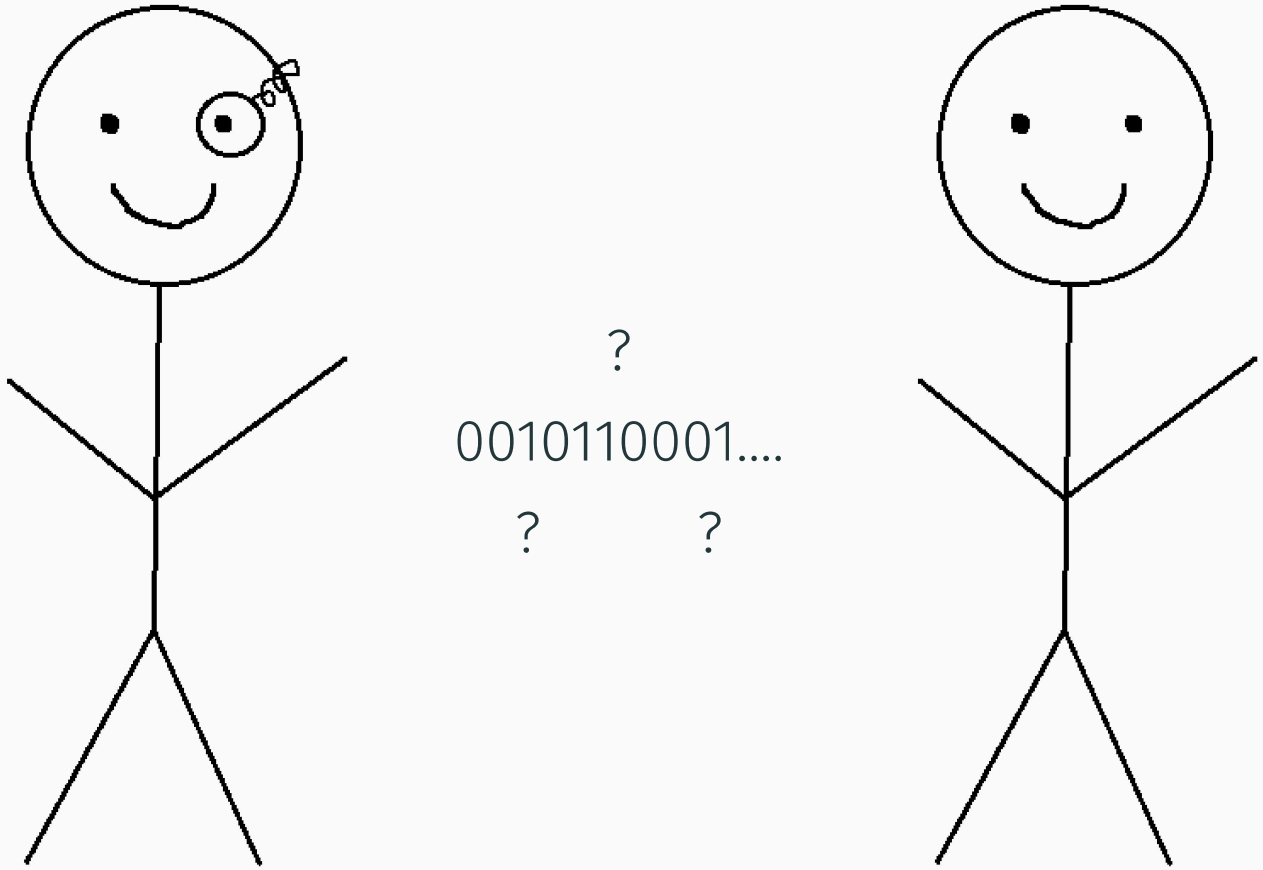


Alice

Intrinsic randomness

- Outcomes are **private** i.e. they cannot be predicted by *anyone*.



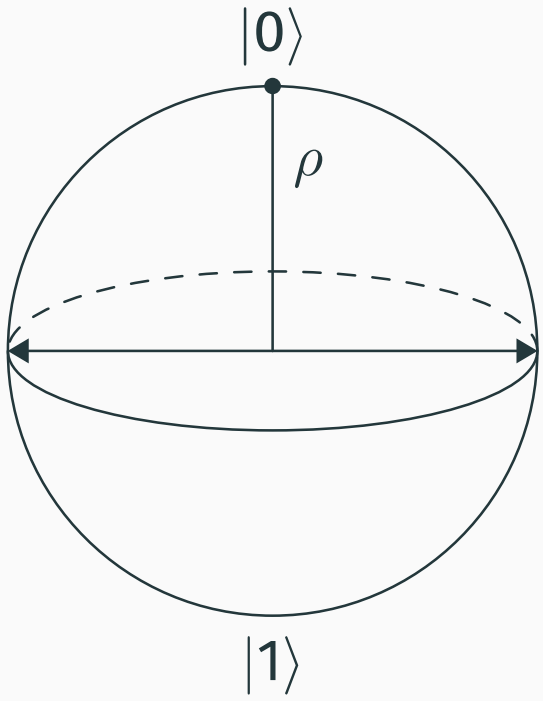
Intrinsic randomness from superposition

Quantum state	ρ	$\rho \geq 0\,, \qquad \mathbf{tr}\,\rho = 1$
Measurement	$\{M_i\}$	$M_i \geq 0\,, \quad \sum_i M_i = I$

Born rule : probability of outcome $x = \mathbf{tr}\,(M_x\,\rho)$

- **Pure** states: $\rho = |\psi\rangle\langle\psi|$
- Rank-one **projective** measurements: $M_i = |m_i\rangle\langle m_i|$

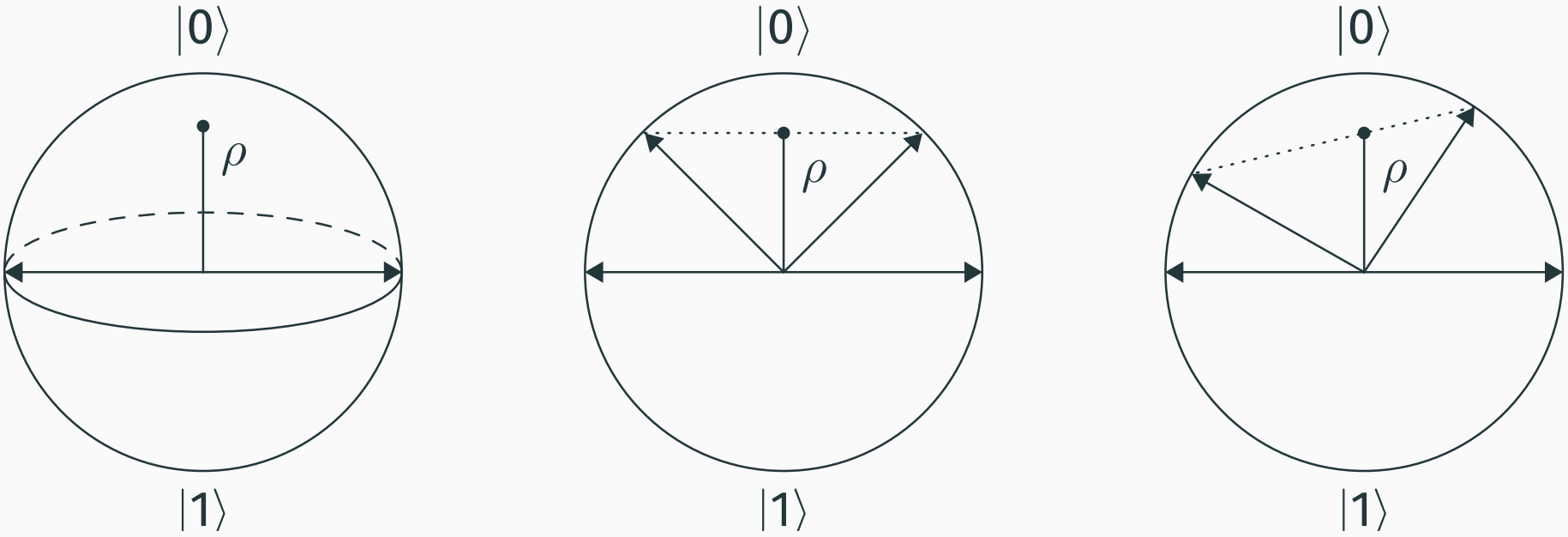
Uniform distribution **and** perfect intrinsic randomness!



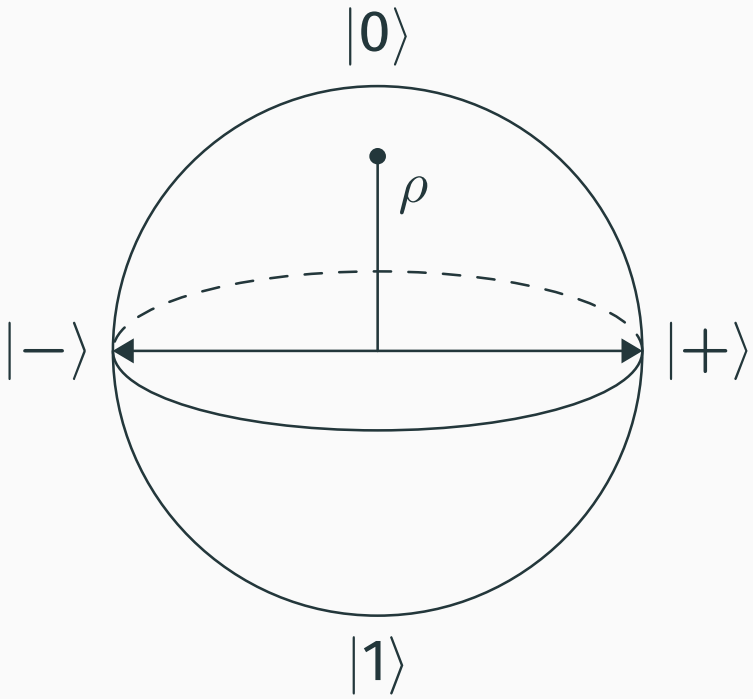
Mixed quantum states

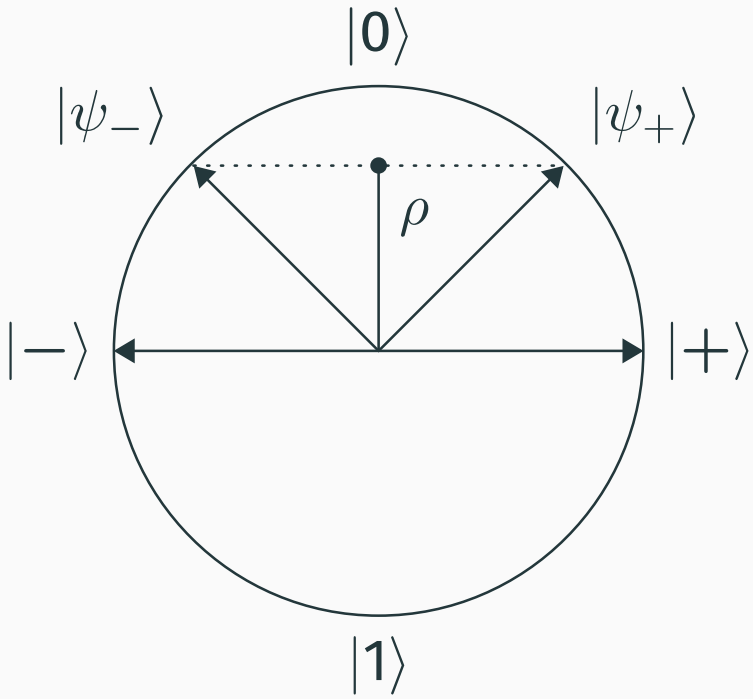
- A **mixed** quantum state can be represented as a *probabilistic mixture* of other quantum states,

$$\rho = \sum_i p_i \, \rho_i \,, \quad \sum_i p_i = 1 \,.$$



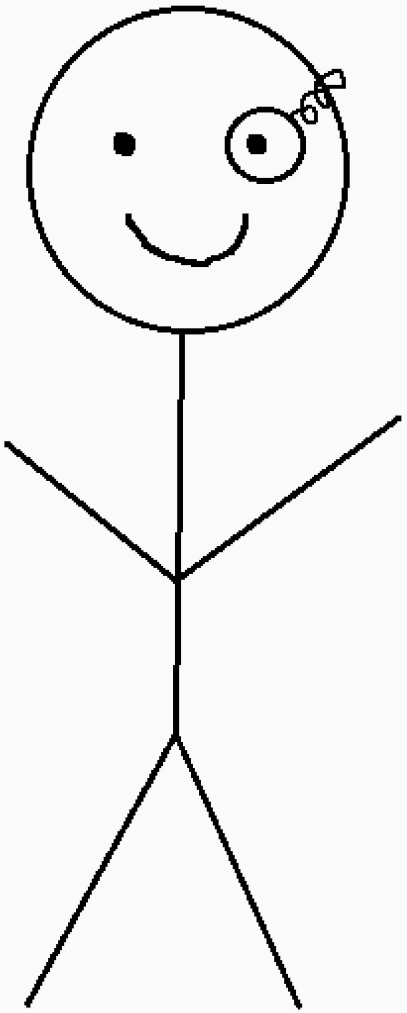
A quantum eavesdropper





- You have a **trusted** quantum state ρ from an **untrusted** source.
- The **actual state** $|\psi_+\rangle$ or $|\psi_-\rangle$ in every round could be known by an eavesdropper.

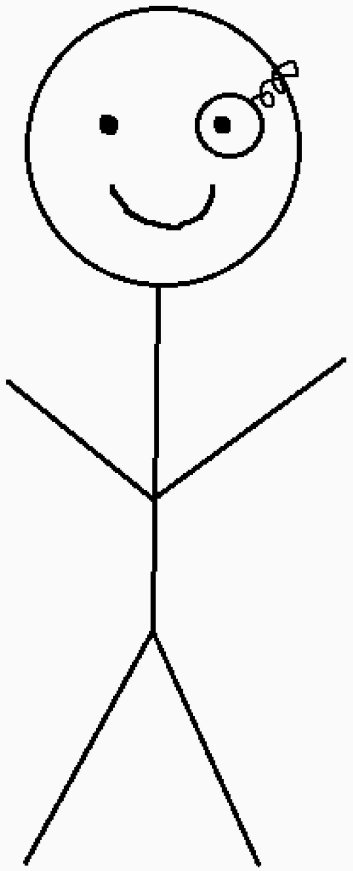
A quantum eavesdropper



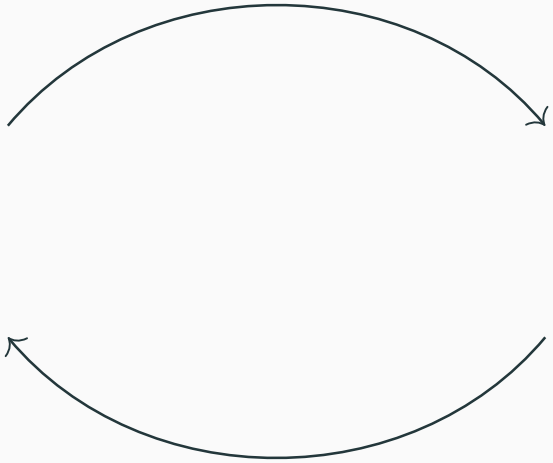
- Eve knows the actual state in each round.
- Assume the **decomposition** $\rho = \sum_i p_i \rho_i$ is optimal for Eve.
- It's equivalent[a] to say Eve *steers the states* using a purification of ρ.

[a] L. P. Hughston, R. Jozsa, and W. K. Wootters, A complete classification of quantum ensembles having a given density matrix, *Physics Letters A* 183, 14–18, 1993

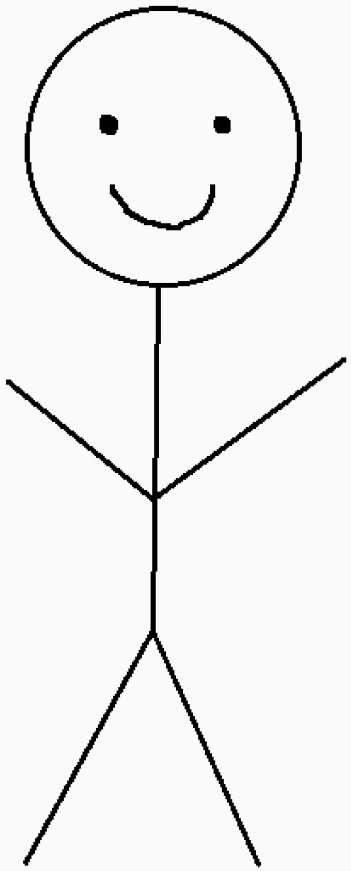
Steering and state discrimination



Eve steers ρ_i with probability p_i



Eve discriminates the states
$\{\langle m_i|\Psi\rangle^{AE}\}$



Alice receives
$\rho = \sum_i p_i \rho_i$

Alice measures
$\{|m_i\rangle\}$

Eve

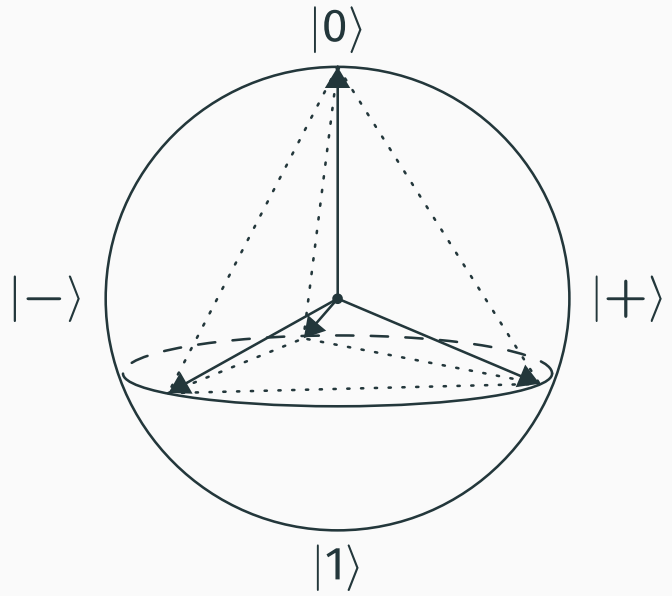
Alice

$|\Psi\rangle^{AE}$

Alice's measurement

To perform a *more general* measurement (**POVM**), we would need

- An additional quantum state
- An additional source of randomness



We also show that **coarse-graining** does not increase randomness.

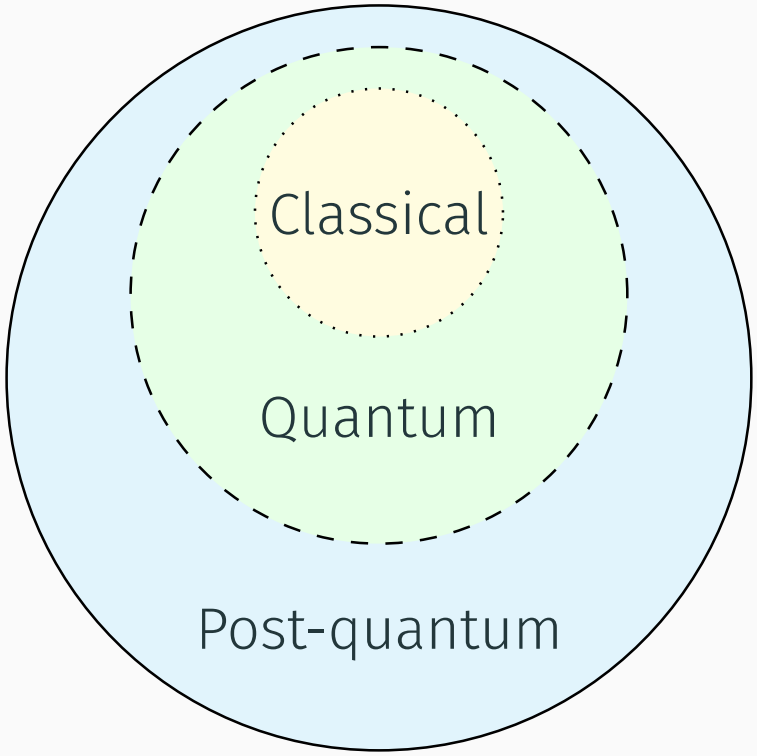
$$\{\,|0\rangle\langle 0|\,,\,|1\rangle\langle 1|\,,\,|2\rangle\langle 2|\,\} \longrightarrow \{\,|0\rangle\langle 0| + |1\rangle\langle 1|\,,\,|2\rangle\langle 2|\,\}$$

Motivations

Why find the maximal intrinsic randomness of ρ?

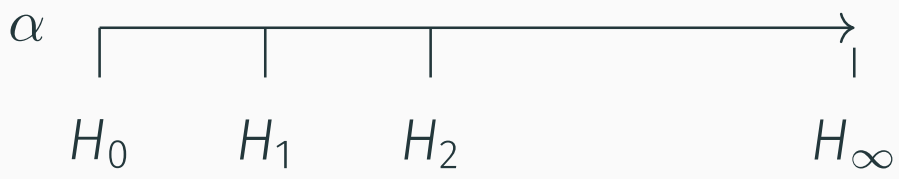
- Bound the functionality of **quantum random number generators** (QRNGs).

- Quantifying quantumness?



Quantifying uncertainty

- We could use the family* of **Rényi entropies** H_α.



- For randomness, we use the **conditional quantum** entropies† and the *classical-quantum* state

$$\rho_{XE} = \sum_{x \in \mathcal{X}} p_x \, |x\rangle\langle x| \otimes \rho_E^x \,,$$

where $\mathcal{X}$ is the set of measurement outcomes.

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*A. Rényi, On measures of information and entropy, *Proc. Symp. on Math., Stat. and Probability*, 547–561, Berkeley, 1961.

†M. Müller-Lennert, F. Dupuis, O. Szehr, S. Fehr, and M. Tomamichel, On quantum Rényi entropies: A new generalization and some properties. *Journal of Mathematical Physics*, 54(12), December 2013.

Conditional min-entropy

$H_{\min}$ has the operational interpretation[a]

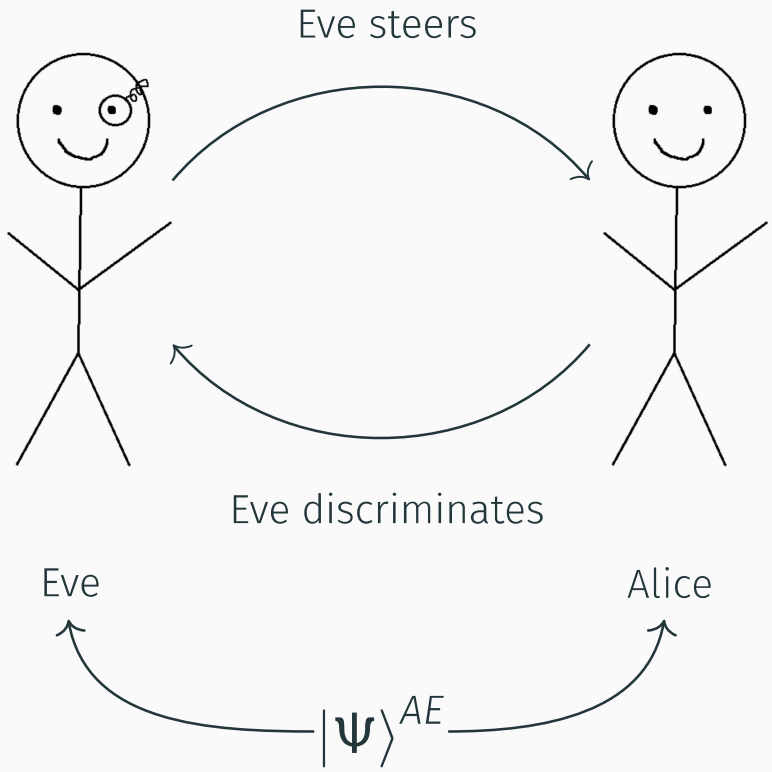
$$H_{\min} = -\log_2 P_{\text{guess}} \,,$$

where P_{guess} is the **guessing probability**

$$P_{\text{guess}} = \max_{\{p_i\},\{\rho_i\}} \sum_i p_i \, \mathrm{tr}(M_i \, \rho_i) \,,$$

subject to

$$\sum_i p_i \, \rho_i = \rho \,.$$



[a] R. Konig, R. Renner, and C. Schaffner, The operational meaning of min- and max-entropy, *IEEE Transactions on Information Theory* 55, 4337, 2009

Conditional von Neumann entropy

- The von Neumann entropy is

$$S(\rho) = -\operatorname{tr}\left(\rho \log_2 \rho\right) .$$

- Given a bipartite state ρ_{AE}, the *conditional* entropy is

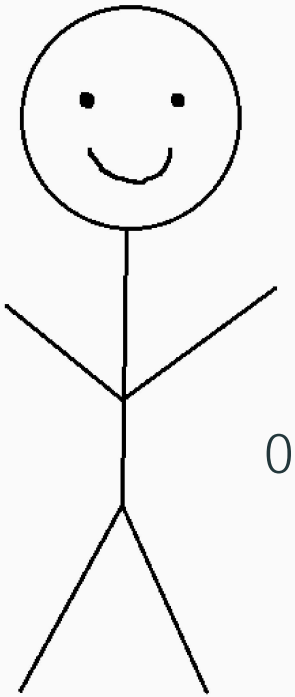
$$H(A|E) = S(\rho_{AE}) - S(\rho_E) .$$

- Interpretation as the **communication cost** for *quantum state merging*[a].

John von Neumann

[a] M. Horodecki, J. Oppenheim, A. Winter, Partial quantum information, *Nature* 436 (7051): 673–676, 2005

Conditional max-entropy



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- For a tripartite state ρ_{ABC}, we have[a]

$$H_{\min}(A|C) = -H_{\max}(A|B)\,.$$

- $H_{\max}$ has an operational interpretation[b] as the **security of a secret key** created from the measurement outcomes.

---

[a] R. Renner, Security of quantum key distribution, PhD thesis, 2006

[b] R. Konig, R. Renner, and C. Schaffner, The operational meaning of min- and max-entropy, *IEEE Transactions on Information Theory* 55, 4337, 2009

Important relation!

$$H_{\min} \leq H \leq H_{\max}$$

- $H_{\min}$ and $H_{\max}$ bound H in the **single-shot** scenario (where Alice makes the measurement only once).

Summary of results

$H^*_{\min}$	$\log_2 d - 2 \log_2 \operatorname{tr} \sqrt{\rho}$
H^*	$\log_2 d - S(\rho)$
$H^*_{\max}$	$\log_2 d + \log_2 \lambda_{\max}(\rho)$

Bounds on maximal conditional entropies.

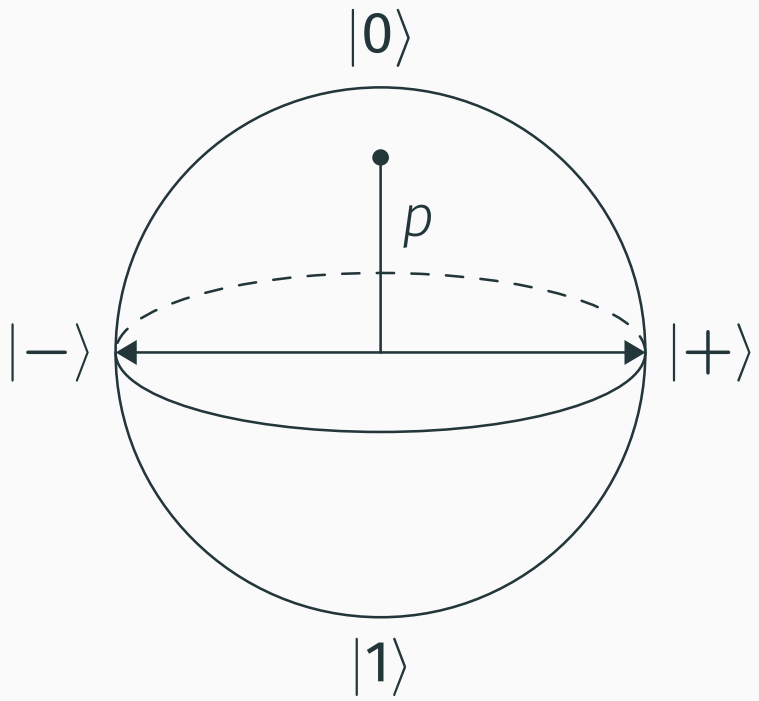
$H^*_{\min}$	$\langle m_i \vert \sqrt{\rho} \vert m_i \rangle = \frac{1}{d} \operatorname{tr} \sqrt{\rho}$	for all $i = 1, \ldots, d$
H^*	$\langle m_i \vert \rho \vert m_i \rangle = \frac{1}{d}$	for all $i = 1, \ldots, d$
$H^*_{\max}$	$\vert \langle m_i \vert u_{\max} \rangle \vert^2 = \frac{1}{d}$	for all $i = 1, \ldots, d$

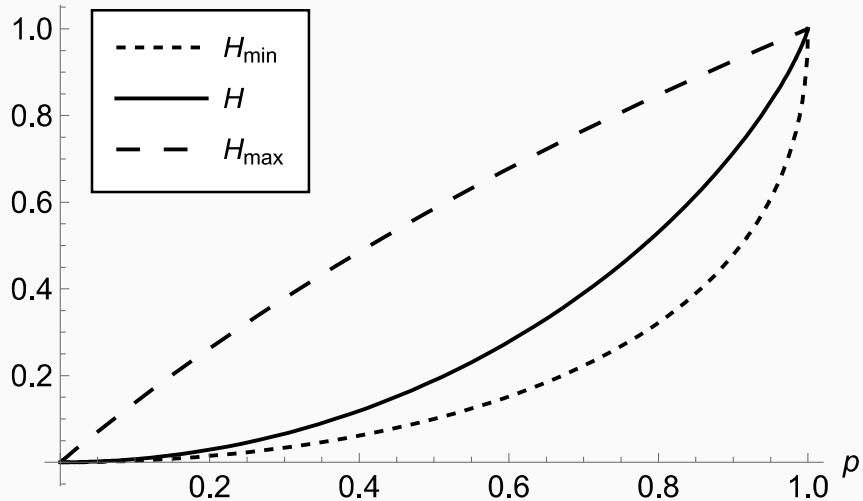
Necessary and sufficient conditions for Alice's optimal measurement.

Qubit results

Consider a **noisy** qubit state

$$\rho = p\,|0\rangle\langle 0| + (1-p)\,\frac{I}{2}\,.$$

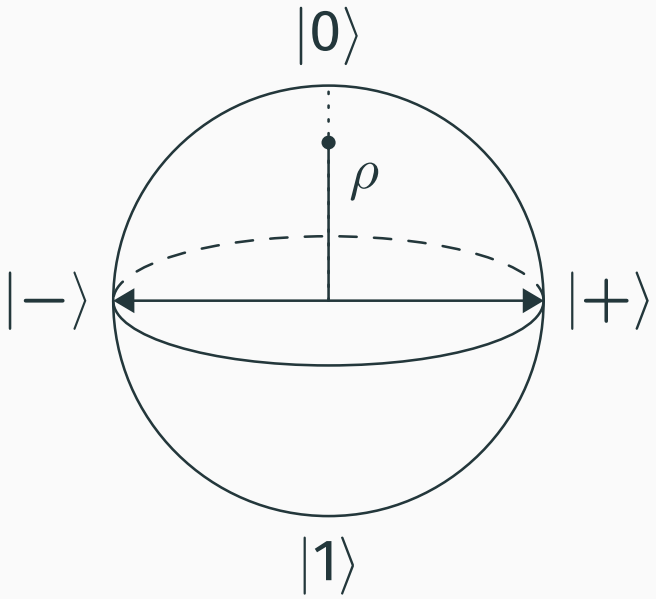




Min-entropy: fidelity

- The **fidelity** $F(\rho, \sigma)$ measures the *distance* between ρ and σ,

$$F(\rho, \sigma) = \left(\operatorname{tr} \sqrt{\sqrt{\rho}\sigma\sqrt{\rho}} \right)^2 .$$



It was shown[‡] that

$$P_{\text{guess}}(\rho, \{M_i\}) = \max_{\sigma} F(\rho, \sigma) ,$$

where σ is diagonal in $\{|m_i\rangle\}$. Then

$$P_{\text{guess}}(\rho, \{M_i\}) \geq F(\rho, \mathbb{I}/d) = \frac{1}{d} (\operatorname{tr} \sqrt{\rho})^2 .$$

[‡] P. J. Coles, Unification of different views of decoherence and discord, *Physical Review A* 85, 2012

Min-entropy: semidefinite programming

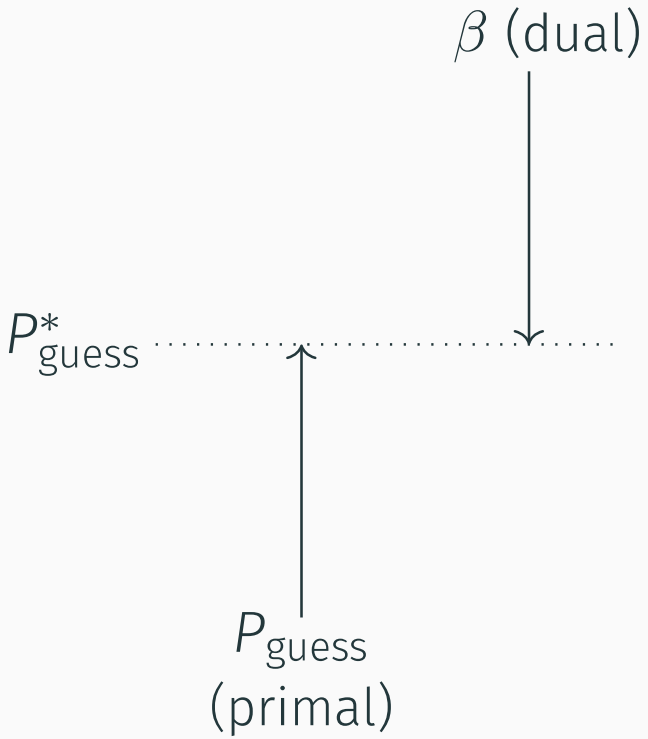
- Semidefinite programming (SDP) involves **optimisation** subject to *equality* ($X = Y$) and *inequality* ($X \succcurlyeq Y$) constraints.

- Every **primal** problem P has a **dual** problem β, with

$$\max P = \min \beta \, .$$

- Since P^*_{guess} is an SDP, we can find

$$P^*_{\text{guess}} \leq \beta \, .$$

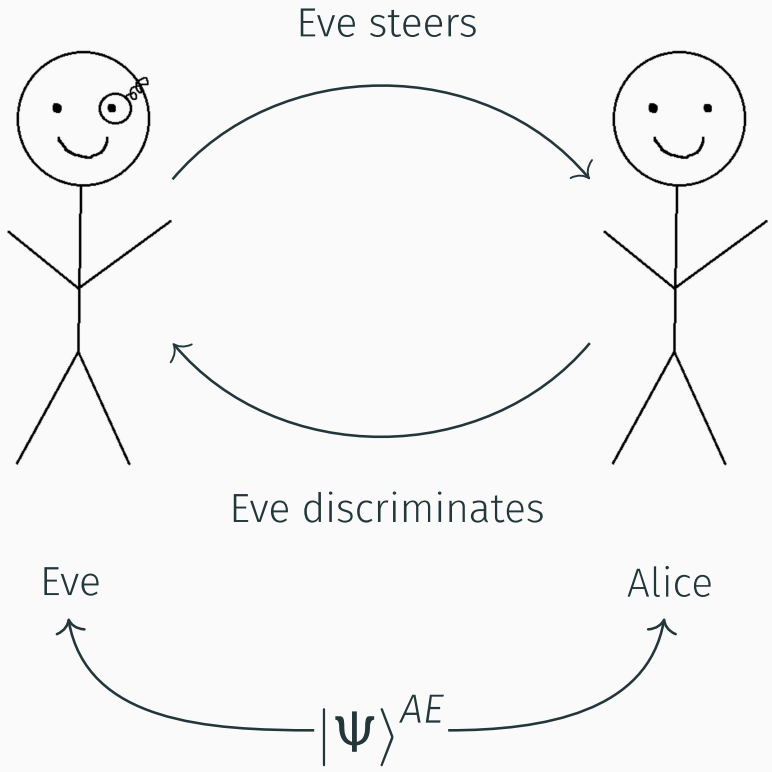


Min-entropy: square root measurements

- Given an ensemble $\{p_i,\ \rho_i\}$, a *square root measurement*[a] (SRM) satisfies

$$M_i = p_i\, \rho^{-\frac{1}{2}}\, \rho_i\, \rho^{-\frac{1}{2}}\ .$$

- Optimal at **state discrimination** for many ensembles.



[a] P. Hausladen and W. K. Wootters, A 'pretty good' measurement for distinguishing quantum states, *Journal of Modern Optics* 41, 2385 (1994).

If Alice's measurement is optimal,

- In the steering picture, **Alice** performs a SRM.
- In the state discrimination picture, **Eve** performs a SRM.

Von Neumann entropy: statistical randomness

$H^*_{\min}$	$\langle m_i \vert \sqrt{\rho} \vert m_i \rangle = \frac{1}{d} \operatorname{tr} \sqrt{\rho}$	for all $i = 1, \ldots, d$
H^*	$\langle m_i \vert \rho \vert m_i \rangle = \frac{1}{d}$	for all $i = 1, \ldots, d$
$H^*_{\max}$	$\vert \langle m_i \vert u_{\max} \rangle \vert^2 = \frac{1}{d}$	for all $i = 1, \ldots, d$

The probability of getting outcome i is

$$P(i) = \operatorname{tr}(\rho M_i) = \langle m_i | \rho | m_i \rangle .$$

- Maximal von Neumann entropy gives **perfect statistical randomness**.
- Not necessarily true for the other entropies!

Max-entropy

- Remember that
$$H_{\min}(A|C) = -H_{\max}(A|B)\,.$$

- We find
$$H^*_{\max}(\rho) = \log_2 d + \log_2 \lambda_{\max}(\rho) = \log_2 d - H_\infty(\rho)\,,$$
where $H_\infty(\rho)$ is the **non-conditional** min-entropy of ρ.

Unbiased measurements

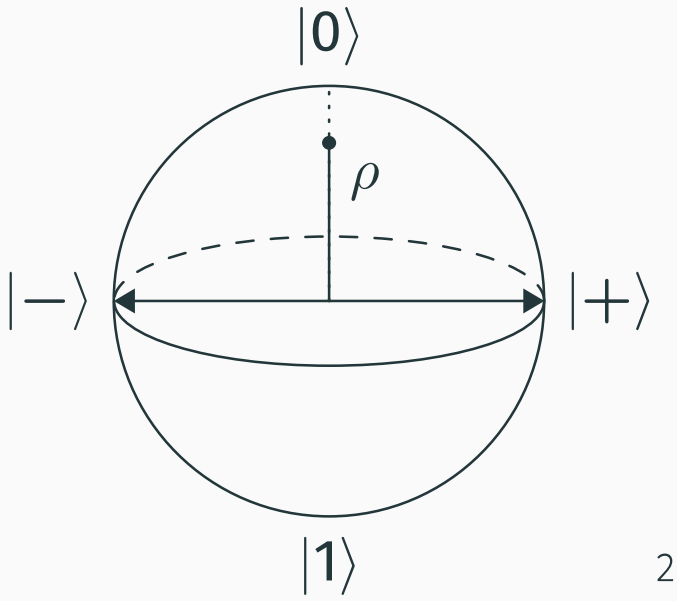
$H^*_{\min}$	$\langle m_i \vert \sqrt{\rho} \vert m_i \rangle = \frac{1}{d} \operatorname{tr} \sqrt{\rho}$	for all $i = 1, \ldots, d$
H^*	$\langle m_i \vert \rho \vert m_i \rangle = \frac{1}{d}$	for all $i = 1, \ldots, d$
$H^*_{\max}$	$\lvert \langle m_i \vert u_{\max} \rangle \rvert^2 = \frac{1}{d}$	for all $i = 1, \ldots, d$

- The bases $\{|m_i\rangle\}$ and $\{|u_j\rangle\}$ are **unbiased** if and only if

$$\left|\langle u_j|m_i\rangle\right|^2 = \frac{1}{d} \quad \text{for all} \quad i, j\,.$$

- If $\{|m_i\rangle\}$ is unbiased to the **diagonal basis** of ρ, we maximise *all three* entropies.

$$\rho = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$



Inequivalent entropies

Consider a qutrit state diagonal in the basis $\{|i\rangle\}$,

$$\rho = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \qquad \lambda_1 \geq \lambda_2 \geq \lambda_3$$

measured in the (**not unbiased**) basis $\{|m_i\rangle\}$, with

$$|m_1\rangle = \sqrt{\frac{2-\gamma_2+\gamma_3}{6}}\,|1\rangle + \sqrt{\frac{2+\gamma_1-\gamma_3}{6}}\,|2\rangle + \sqrt{\frac{2-\gamma_1+\gamma_2}{6}}\,|3\rangle\ ,$$

$$|m_2\rangle = \sqrt{\frac{2-\gamma_2+\gamma_3}{6}}e^{i\theta_1}\,|1\rangle + \sqrt{\frac{2+\gamma_1-\gamma_3}{6}} + \sqrt{\frac{2-\gamma_1+\gamma_2}{6}}e^{i\theta_2}\,|3\rangle\ .$$

- Choose $\gamma_i = \sqrt{\lambda_i}$ for $H^*_{\min}$, $\gamma_i = \lambda_i$ for H^* or $\gamma_2 = \gamma_3$ for ${H_{\max}}^*$.

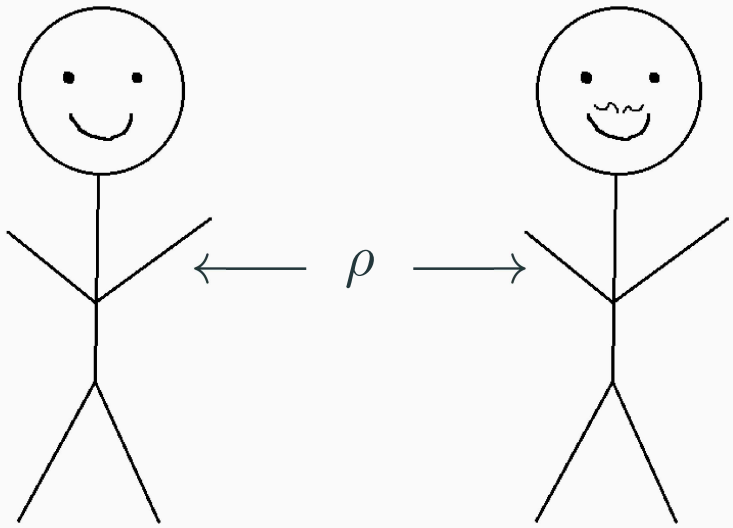
Different measurements maximise different entropies!

Optimal local measurements

Consider a state ρ diagonal in the basis $\{|\psi_i\rangle\}$, where $\omega = e^{\frac{2\pi i}{3}}$.

- There exists **no separable basis** unbiased to $\{|\psi_i\rangle\}$.
- Numerically, we find **optimal local measurements** for all three entropies.

$$\rho = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$$



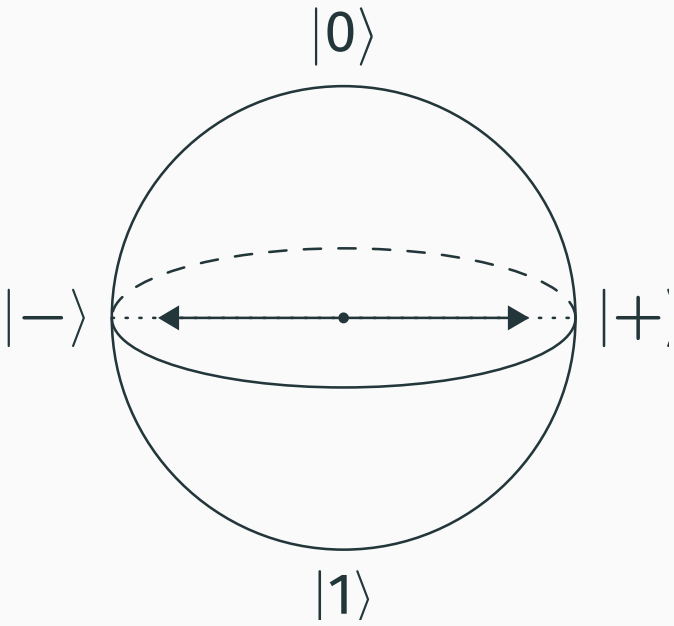
$$|\psi_1\rangle = |00\rangle$$
$$|\psi_2\rangle = \frac{1}{\sqrt{3}}\left(|01\rangle + |10\rangle + |11\rangle\right)$$
$$|\psi_3\rangle = \frac{1}{\sqrt{3}}\left(|01\rangle + \omega\,|10\rangle + \omega^2\,|11\rangle\right)$$
$$|\psi_4\rangle = \frac{1}{\sqrt{3}}\left(|01\rangle + \omega^2\,|10\rangle + \omega\,|11\rangle\right)$$

Limitations

- **Device dependence**: we need to know the complete characterisation of the state ρ and the measurement.

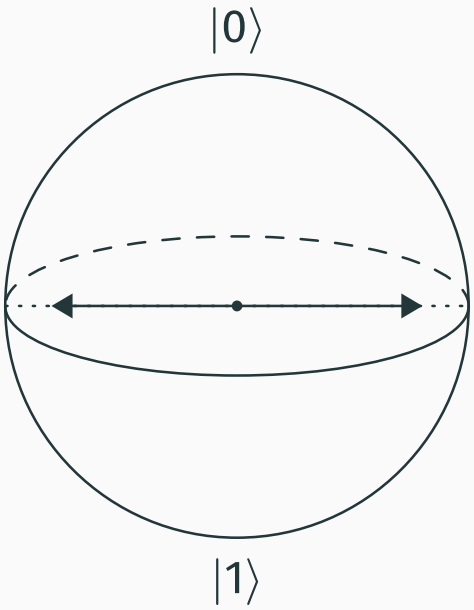
- Does not allow for **noisy** measurements e.g.

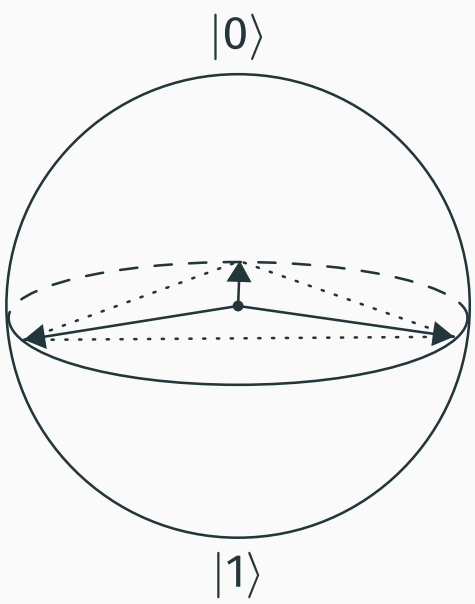
$$M_i = p\,|m_i\rangle\langle m_i| + (1-p)\,\frac{I}{d}\,.$$

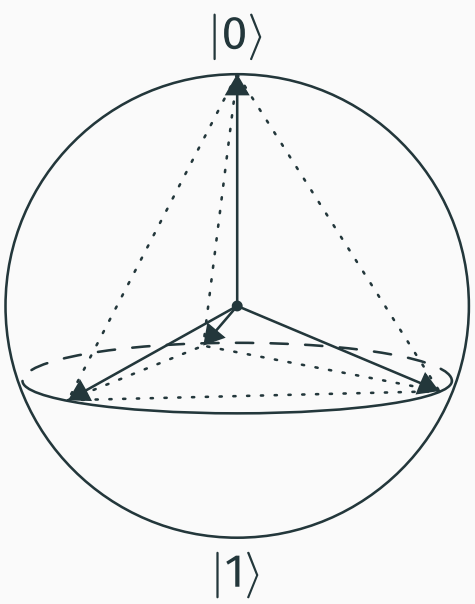


Future work

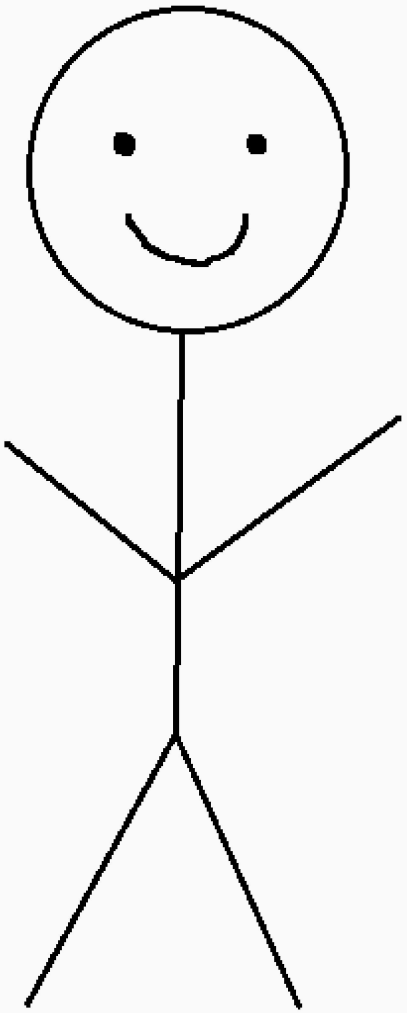
- **Inverse problem**: how much intrinsic randomness can we extract from a quantum *measurement*?







Take-home message



- Quantum randomness has an *intrinsic*, or *private*, quality.
- We can extract

$$H^*_{\min} = \log_2 d - 2 \log_2 \operatorname{tr} \sqrt{\rho}$$
$$H^* = \log_2 d - S(\rho)$$
$$H^*_{\max} = \log_2 d + \log_2 \lambda_{\max}(\rho)$$

  from a **characterised** state ρ.
- If in doubt, use a measurement **unbiased** to the diagonal basis of ρ.

Merci!

Thank you!

Deleted scenes

Eve's guessing probability

- Alice performs a **projective** measurement $\mathcal{M} = \{M_i\}_i$, such that $M_i M_j = M_i \delta_{ij}$.

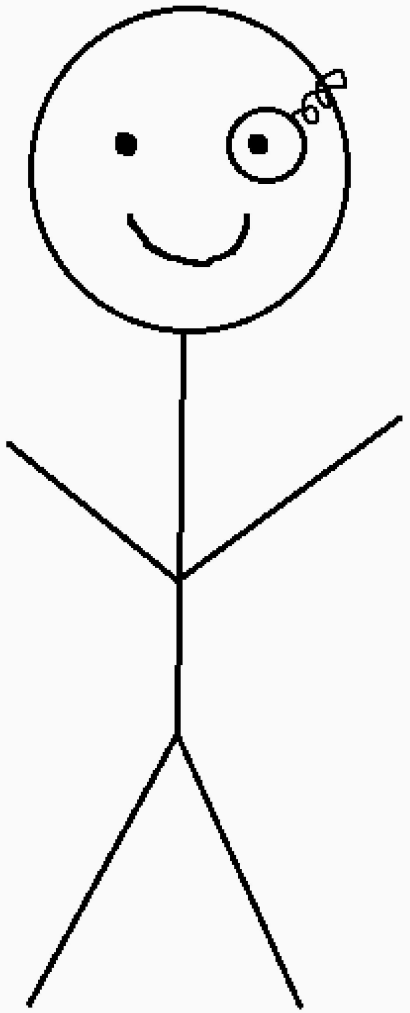
- The probability that Eve correctly guesses Alice's measurement outcomes is a **semidefinite programming (SDP)** problem

$$P_{\text{guess}}(\rho, \mathcal{M}) = \max_{\{p_i,\, \rho_i\}} \sum_i p_i \ \text{tr}(M_i \rho_i)\,,$$

subject to $\ p_i \geq 0\,,\ \sum_i p_i = 1\,,\ \rho_i \geq 0\,,\ \sum_i p_i\, \rho_i = \rho\,.$

- Alice chooses $\mathcal{M}$ to **minimise** Eve's guessing probability,

$$P^*_{\text{guess}}(\rho) = \min_{\mathcal{M}} P_{\text{guess}}(\rho, \mathcal{M})\,.$$



$H_{\min}$ derivation

Previously known[§] that

$$P_{\text{guess}}(\rho, \mathcal{M}) = \max_{\{\sigma \in \mathcal{I}_{\mathcal{M}}\}} F(\rho, \sigma)\,,$$

where F is the **Uhlmann fidelity**

$$F(\rho, \sigma) = \left(\operatorname{tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)^2$$

and $\mathcal{I}_{\mathcal{M}}$ is the set of states **diagonal in the measurement basis** $\{|m_i\rangle\}$. Since $\mathbb{I}/d \in \mathcal{I}_{\mathcal{M}}$ for any $\mathcal{M}$,

$$P_{\text{guess}}(\rho, \mathcal{M}) \geq F(\rho, \mathbb{I}/d) = \frac{1}{d}(\operatorname{tr}\sqrt{\rho})^2\,.$$

[§] P. J. Coles, Unification of different views of decoherence and discord, *Physical Review A* 85, 2012

$H_{\min}$ derivation

The **primal** problem,

$$P_{\text{guess}}(\rho, \mathcal{M}) = \max_{\{p_i,\, \rho_i\}} \sum_i p_i \ \text{tr}(M_i \rho_i)\,,$$

$$\text{subject to } \ p_i \geq 0\,,\ \sum_i p_i = 1\,,\ \rho_i \geq 0\,,\ \sum_i p_i\, \rho_i = \rho\,.$$

has a corresponding **dual problem**,

$$P_{\text{guess}}(\rho, \mathcal{M}) = \min_X \text{tr}\left(X\rho\right), \quad \text{subject to } \ X \geq M_i\,.$$

When the measurement is optimal, we can use $X = \frac{\text{tr}\sqrt{\rho}}{d}\rho^{-\frac{1}{2}}$ to find

$$P_{\text{guess}}(\rho, \mathcal{M}) \leq \frac{1}{d}\left(\text{tr}\sqrt{\rho}\right)^2.$$

so the bound can be reached.

H derivation

- It was shown[¶] that

$$H = S(\rho_{\text{after}}) - S(\rho)\,,$$

where

$$\rho_{\text{after}} = \sum_i \langle m_i|\rho|m_i\rangle\, |m_i\rangle\langle m_i|$$

is Alice's average **post-measurement state**.

- $S(\rho_{\text{after}})$ reaches its maximal value of $\log_2 d$ if and only if $\rho_{\text{after}} = \frac{\mathbb{I}}{d}$, i.e. ρ_{after} is **maximally mixed**, and

$$H^* = \log_2 d - S(\rho)\,.$$

[¶] P. J. Coles, Unification of different views of decoherence and discord, *Physical Review A* 85, 2012

$H_{\max}$ derivation

- We want to find

$$P_{\text{secr}} = d \max_{\sigma} F\left(\rho_{XE}, \frac{I}{d} \otimes \sigma\right) .$$

- For **rank-one projective** measurements,

$$p_{\text{secr}} = \max_{\sigma} \left(\sum_{x} \sqrt{\text{tr}\left(\sigma \, p_x \left|\psi_x^E\right\rangle\!\left\langle\psi_x^E\right|\right)} \right)^2 ,$$

  where $\left|\psi_x^E\right\rangle$ are Eve's local states conditioned on outcome x.

- Using the **Cauchy-Schwartz inequality** and the **SDP** for the largest eigenvalue of a quantum state,

$$p_{\text{secr}} \leq d \max_{\sigma} \text{tr}\left(\sigma\rho\right) = d \, \lambda_{\max}\left(\rho\right) .$$