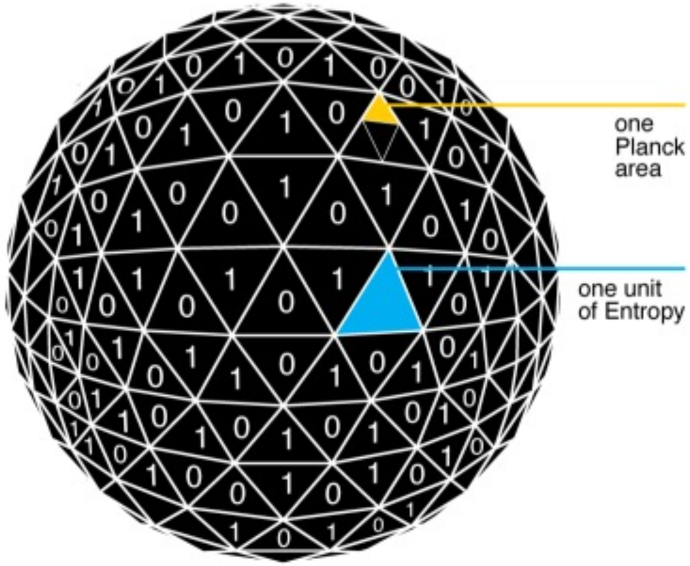
Quantum features from classical entropies



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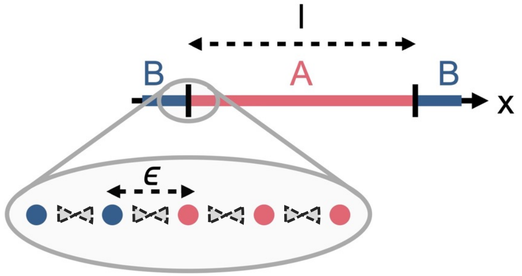
arXiv:2403.12320, arXiv:2404.12321, arXiv:2404.12323

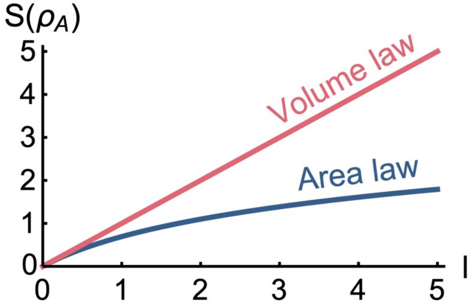
Big picture

- *Entanglement* in subregion

bosons,
fermions,
spins, …

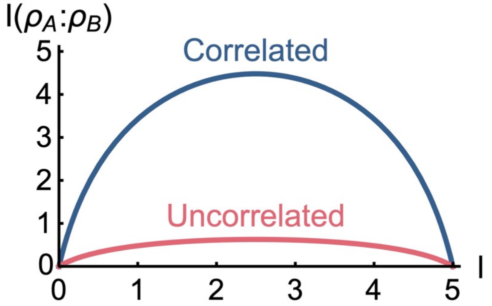


local state $\boldsymbol{\rho}_A = \mathbf{Tr}_B\{\boldsymbol{\rho}\}$



thermal

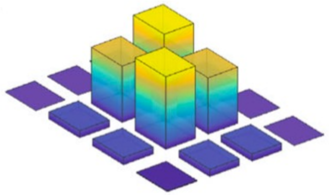
ground,
quenched
states, …

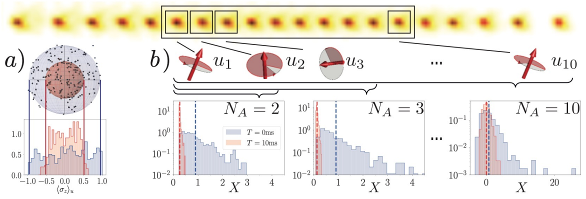


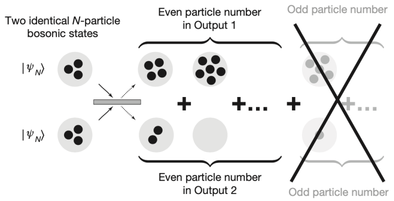
$$S(\boldsymbol{\rho}_A) = -\mathbf{Tr}_A\{\boldsymbol{\rho}_A \ln \boldsymbol{\rho}_A\}$$

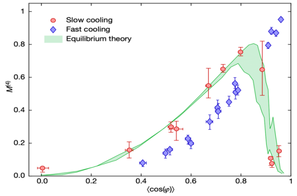
$$I(\boldsymbol{\rho}_A\!:\!\boldsymbol{\rho}_B) = S(\boldsymbol{\rho}_A) + S(\boldsymbol{\rho}_B) - S(\boldsymbol{\rho})$$

- State tomography *or* clever readout *or* correlators









→ Quantum entropies $S(\boldsymbol{\rho}_A)$ and local state $\boldsymbol{\rho}_A$ *needed*?

Outline

Theory

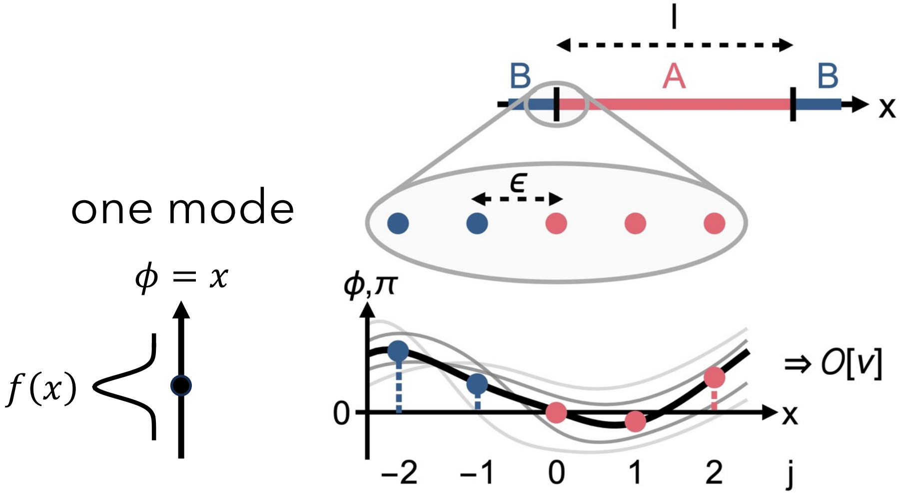
- Subtracted classical entropies
- Scalar field

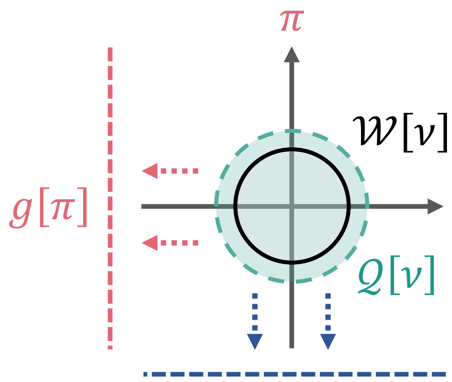
Experimental perspective

- Bose-Einstein condensate
- Area law → Volume law

Measurement distributions

- Fundamental fields $\nu = (\phi, \pi)$; $\boldsymbol{\phi}(x)|\phi\rangle = \phi(x)|\phi\rangle$; $[\boldsymbol{\phi}(x), \boldsymbol{\pi}(x')] = i\delta(x - x')$





$\mathcal{W}[\nu] = \int \frac{\mathcal{D}\phi'}{\pi} \langle \phi - \phi' | \boldsymbol{\rho} | \phi + \phi' \rangle e^{2i \int \mathrm{d}x\, \pi(x) \phi'(x)}$ Wigner

$\mathcal{Q}[\nu] = \int \mathcal{D}\nu'\, \mathcal{W}[\nu']\, \bar{\mathcal{W}}(\nu - \nu')$ Husimi

$f[\phi] = \langle \phi | \boldsymbol{\rho} | \phi \rangle = \int \mathcal{D}\pi\, \mathcal{W}[\nu]$ Marignals

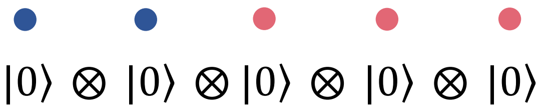
- Measurement distributions $\mathcal{O}[\nu]$

→ Local distributions $\mathcal{O}_A[\nu_A] = \int \mathcal{D}\nu_B\, \mathcal{O}[\nu]$

Classical entropy & uncertainty

- Local classical Rényi entropy

Vacuum



$|0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle$

$$S_r[\mathcal{O}_A] = \frac{1}{1-r}\ln\left[\int \mathcal{D}v_A\, \mathcal{O}_A^r\right] \geq S_r[\bar{\mathcal{O}}_A] \sim \frac{l}{\epsilon}$$

Entropic uncertainty

Volume law

- Subtracted classical entropy

$$\Delta S_r[\mathcal{O}_A] = S_r[\mathcal{O}_A] - S_r[\bar{\mathcal{O}}_A] \;\sim\; \text{entanglement}$$

$\boldsymbol{\rho}$ separable ⇒

$$\Delta S_r[\mathcal{O}_A|\mathcal{O}_B] = \Delta S_r[\mathcal{O}] - \Delta S_r[\mathcal{O}_A] \geq 0$$

- Classical Rényi mutual information

$$I_r[\mathcal{O}_A : \mathcal{O}_B] = S_r[\mathcal{O}_A] + S_r[\mathcal{O}_B] - S_r[\mathcal{O}] \;\sim\; \text{correlations}$$

→ Let's check these out!

Scalar quantum field

- Hamiltonian

$$\boldsymbol{H} = \int \mathrm{d}x \, [\boldsymbol{\pi}^2 + (\partial_x \boldsymbol{\phi})^2 + m^2 \boldsymbol{\phi}^2]$$

- Typical local distribution

$$\mathcal{O}_A[\nu_A] = \underbrace{\frac{1}{Z_A^{\mathcal{O}}} e^{-\frac{1}{2}\int \mathrm{d}x\mathrm{d}x' \nu_A^T(x) (\gamma_A^{\mathcal{O}})^{-1}(x,x') \nu_A(x')}}_{\text{Gaussian}} \times \underbrace{\kappa_A^{\mathcal{O}}[\nu_A]}_{\text{polynomial}}$$

$$\gamma_A^{\mathcal{O}} = \begin{pmatrix} \langle \boldsymbol{\phi}_A \boldsymbol{\phi}_A \rangle_{\mathcal{O}} & \langle \boldsymbol{\phi}_A \boldsymbol{\pi}_A + \boldsymbol{\pi}_A \boldsymbol{\phi}_A \rangle_{\mathcal{O}} \\ \langle \boldsymbol{\phi}_A \boldsymbol{\pi}_A + \boldsymbol{\pi}_A \boldsymbol{\phi}_A \rangle_{\mathcal{O}} & \langle \boldsymbol{\pi}_A \boldsymbol{\pi}_A \rangle_{\mathcal{O}} \end{pmatrix}$$

General formulae

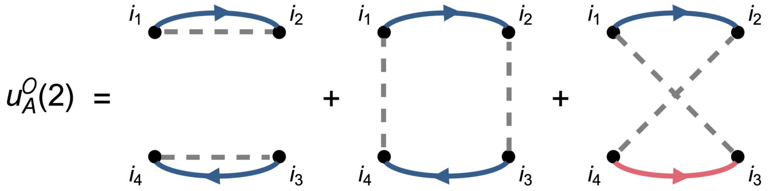
- Classical Rényi entropy

$$S_r[\mathcal{O}_A] = \underbrace{\frac{1}{2}\ln\det\left(2\pi\gamma_A^{\mathcal{O}}\right) + \overset{\text{Volume law}}{\boxed{\frac{1(2)}{2}\,\frac{\ln r}{r-1}\,\frac{l}{\epsilon}}}}_{\text{Gaussian}} + \underbrace{\frac{\ln U_A^{\mathcal{O}}}{1-r}}_{\text{non-Gaussian}}$$

$$U_A^{\mathcal{O}}(r) = (\kappa_A^{\mathcal{O}}[\partial_\zeta])^r\, e^{\frac{1}{2r}\int \mathrm{d}x\mathrm{d}x'\zeta^T(x)\gamma_A^{\mathcal{O}}(x,x')\zeta(x')}\,\Big|_{\zeta=0}$$

- Subtracted classical Rényi entropy

$$\Delta S_r[\mathcal{O}_A] = \frac{1}{2}\ln\det\left[\gamma_A^{\mathcal{O}}\left(\bar{\gamma}_A^{\mathcal{O}}\right)^{-1}\right] + \frac{\ln U_A^{\mathcal{O}}}{1-r}$$



- Classical Rényi mutual information

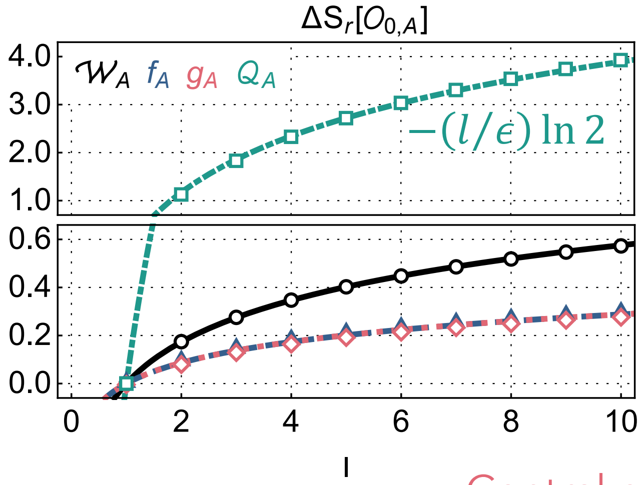
$$I_r[\mathcal{O}_A\!:\mathcal{O}_B] = \frac{1}{2}\ln\frac{\det\left(\gamma_A^{\mathcal{O}}\right)\det\left(\gamma_B^{\mathcal{O}}\right)}{\det\left(\gamma^{\mathcal{O}}\right)} + \frac{1}{1-r}\ln\frac{U_A^{\mathcal{O}}U_B^{\mathcal{O}}}{U^{\mathcal{O}}}$$

Area laws

- Gaussian

$$S_2(\boldsymbol{\rho}_A) = \Delta S_r[\mathcal{W}_A]$$

$$\underset{\mathcal{W} = f\times g}{=} \Delta S_r[f_A] + \Delta S_r[g_A]$$

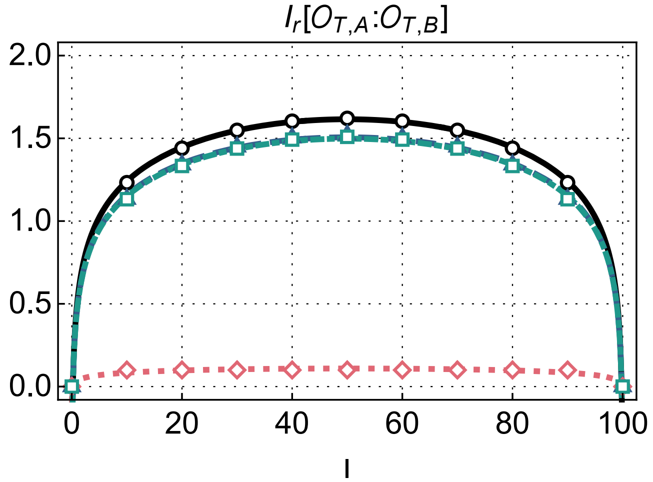


Central charge

$$\Delta S_r[\mathcal{O}_A] \propto c \ln\frac{l}{\epsilon}$$

- Thermal, *local* theory

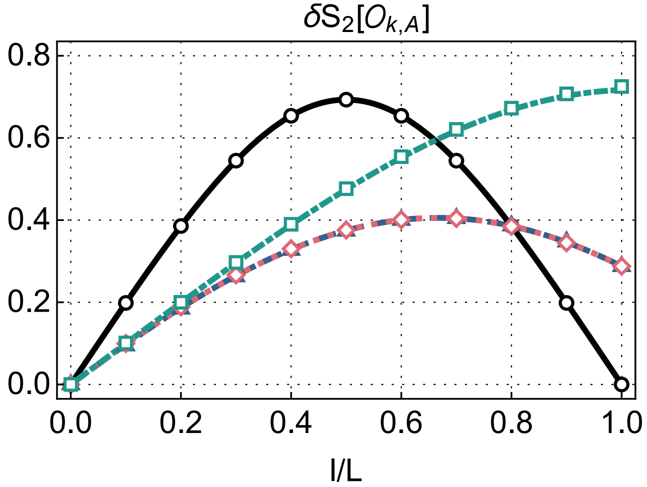
$$\boldsymbol{\rho} = \frac{1}{Z} e^{-\boldsymbol{H}/T}$$



$$I_r\left[\mathcal{O}_{T,A}\colon \mathcal{O}_{T,B}\right] \leq a\,|\partial A|$$

- Particles, high energy

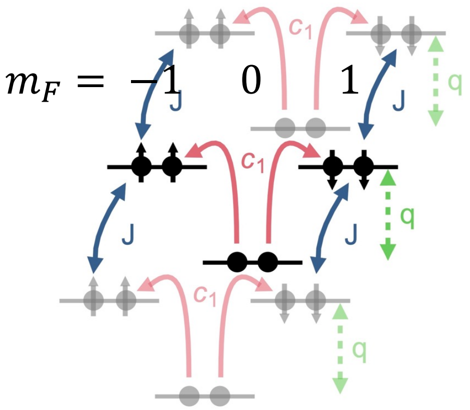
$$\omega(p) = \sqrt{m^2 + p^2} \gg \frac{1}{l}, \frac{1}{L-l}$$



$$\delta S_r\left[\mathcal{O}_{k,A}\right] = \frac{1}{1-r}\ln\left[1 + \sum_{i=1}^{r} a_{r,i}\left(\frac{l}{L}\right)^i\right]$$

Spin-1 Bose-Einstein condensate

- Dynamics: Quench



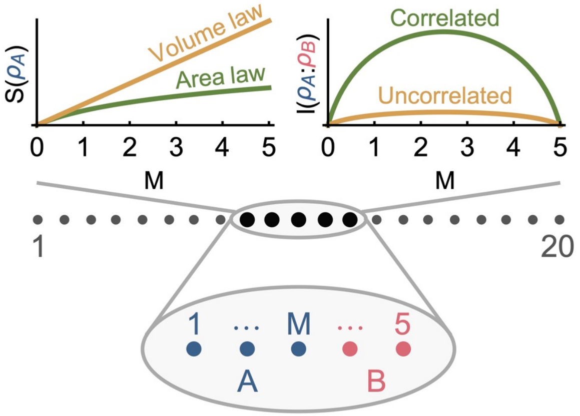
$$\boldsymbol{H} = \sum_{j=1}^{20} q\left(\boldsymbol{N}_1^j + \boldsymbol{N}_{-1}^j\right) + c_0 \boldsymbol{N}^j\left(\boldsymbol{N}^j - \mathbb{I}\right)$$

$$+c_1\left[\left(\boldsymbol{N}_0^j - \tfrac{1}{2}\mathbb{I}\right)\left(\boldsymbol{N}_1^j + \boldsymbol{N}_{-1}^j\right)\right.$$

$$\left.+\boldsymbol{a}_0^{j\dagger}\boldsymbol{a}_0^{j\dagger}\boldsymbol{a}_1^j\boldsymbol{a}_{-1}^j + h.c.\right]$$

$$-J\sum_{j=1}^{19}\sum_{m_F=\pm1}\left(\boldsymbol{a}_{m_F}^{j\dagger}\boldsymbol{a}_{m_F}^{j+1} + h.c.\right)$$

- Readout: Spins

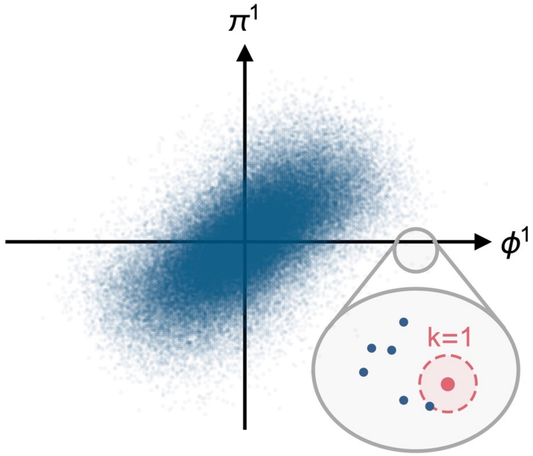


Spin operators

$$\boldsymbol{\phi}^j \equiv \frac{1}{2\sqrt{n}}\left[\boldsymbol{a}_0^{j\dagger}\left(\boldsymbol{a}_1^j + \boldsymbol{a}_{-1}^j\right) + h.c.\right]$$

$$\boldsymbol{\pi}^j \equiv \frac{-i}{2\sqrt{n}}\left[\boldsymbol{a}_0^{j\dagger}\left(\boldsymbol{a}_1^j + \boldsymbol{a}_{-1}^j\right) - h.c.\right]$$

- Entropy estimation: kNN



$$\hat{S}(k, N_s) = g(k, N_s, d) + \frac{d}{N_s}\sum_{i=1}^{N_s}\ln \epsilon^i(k)$$

→ Asymptotically unbiased

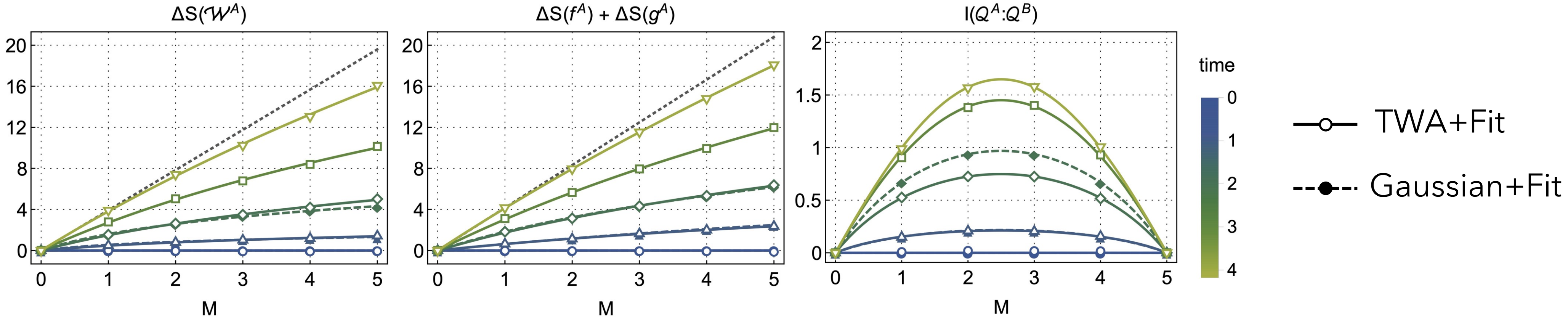
→ Entropy estimated *directly* from data

→ *No* assumptions on $\boldsymbol{\rho}_A$ or $\mathcal{O}_A$

Area law

Subtracted classical entropies | Scalar field | BEC | Area law → Volume law

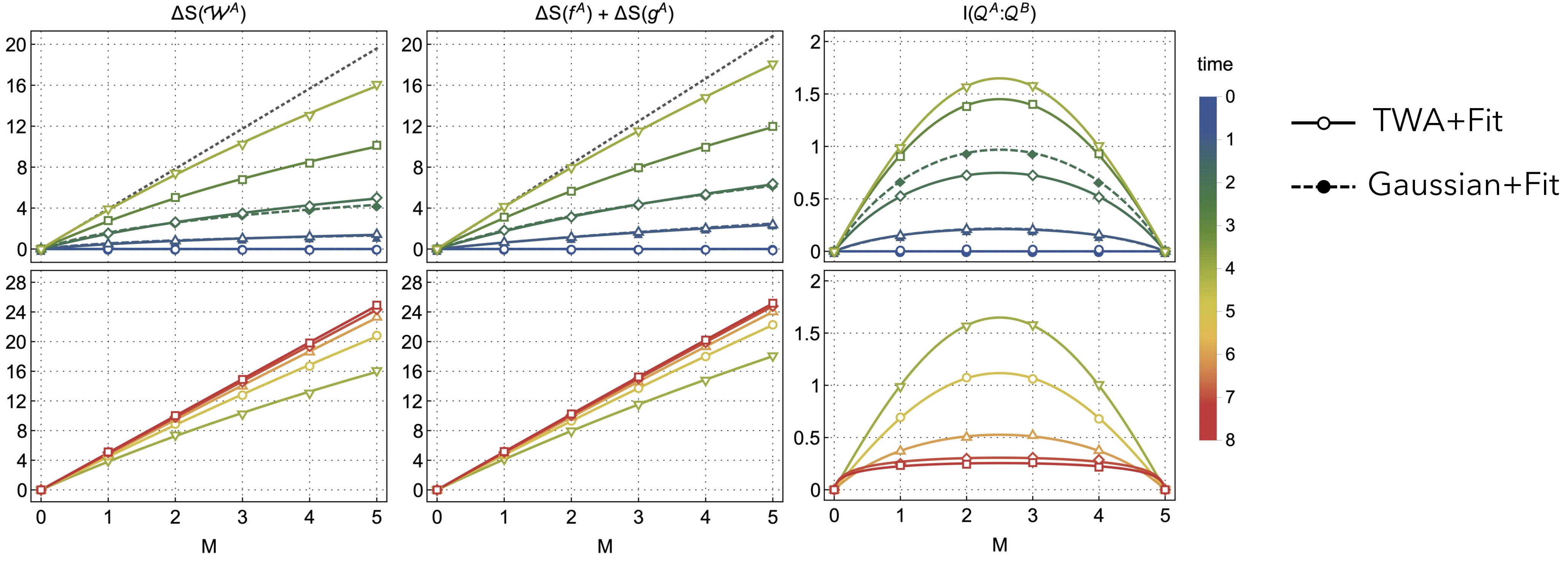
- Early times



→ Area laws signal build-up of quantum correlations

Area law → Volume law

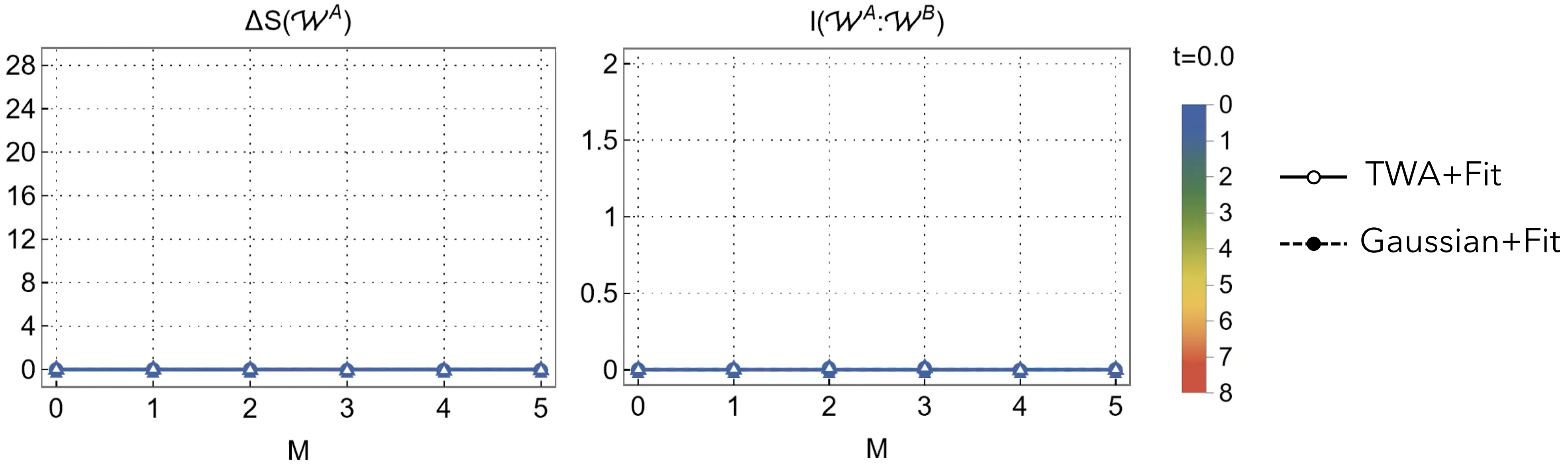
- Early times + late times



→ Volume laws reveal local thermalization, incline = $1/T$

Area law → Volume law

- Full time evolution of $\mathcal{W}$-quantities



Summary

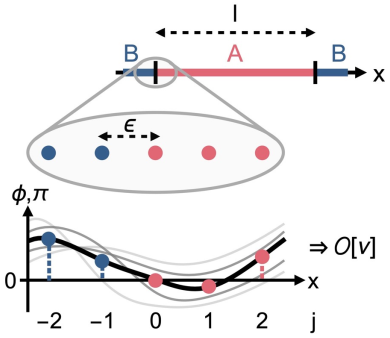
Theory

- Subtracted classical entropies

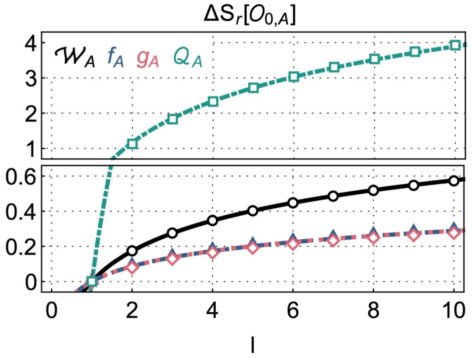
$$\Delta S_r[\mathcal{O}_A] = S_r[\mathcal{O}_A] - S_r[\bar{\mathcal{O}}_A]$$

$$I_r[\mathcal{O}_A\!:\mathcal{O}_B] = S_r[\mathcal{O}_A] + S_r[\mathcal{O}_B] - S_r[\mathcal{O}]$$

- Scalar field



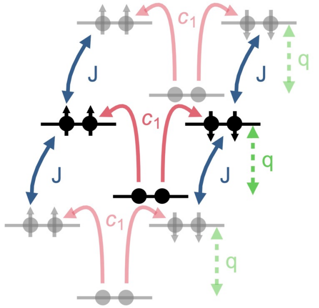
Distributions $\mathcal{O}[v]$

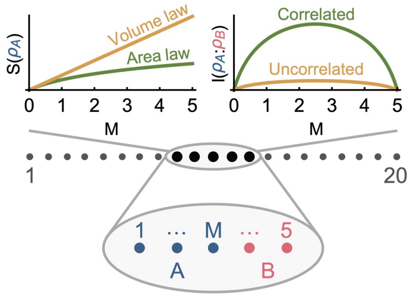


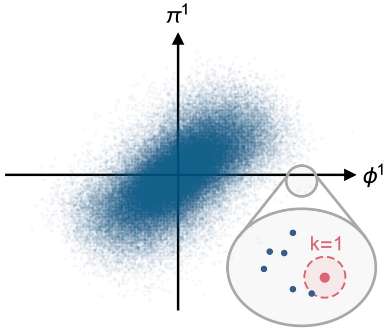
$$\Delta S_r[\mathcal{O}_A] \propto c \ln \frac{l}{\epsilon}$$

Experimental perspective

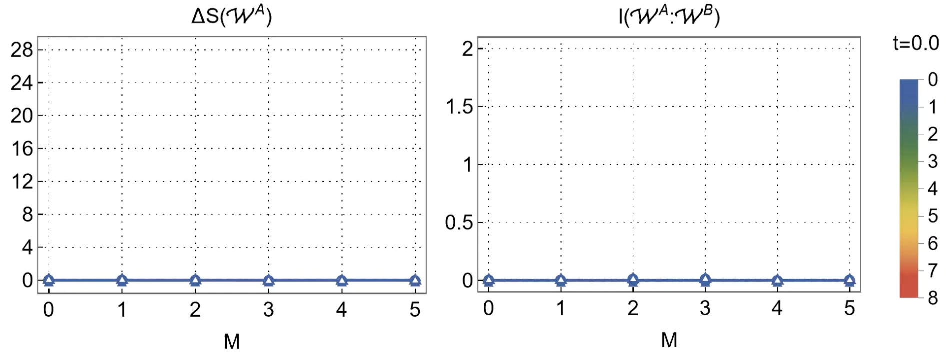
- Spin-1 BEC





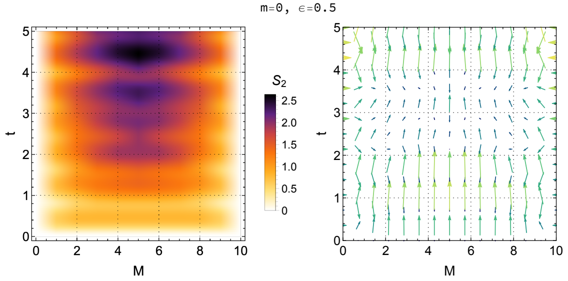


- Area law → Volume law

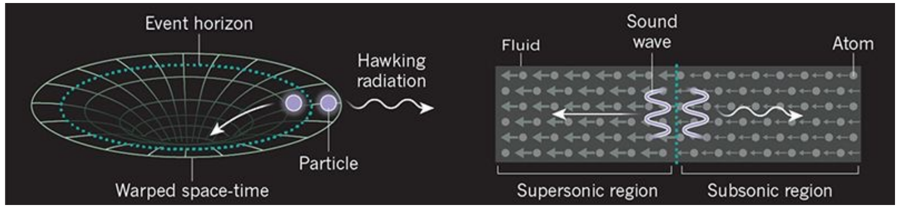


Outlook

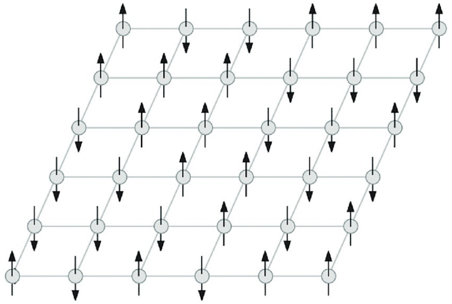
- Classical entropies *should* solve *most* of their quantum analog's problems



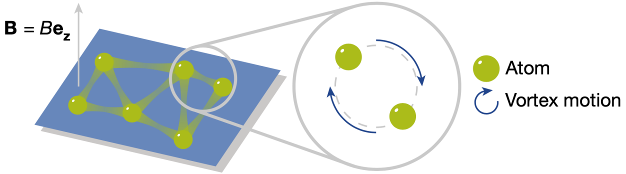
Jena Information flow/scrambling,
Lieb-Robinson bounds for MIs



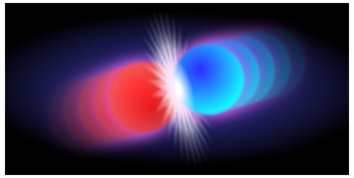
Entropy in analog gravity:
Black holes, expanding spacetime, …



Higher-order correlations,
Spins, QPTs, MBL, Disorder,
Symmetry-resolved entropies



Topological EE,
Central charges, Bruxelles
Chern numbers,
Bulk ↔ Edges



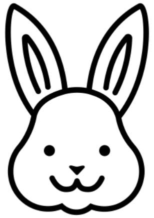
Tunnel-coupled BEC:
Sine-Gordon model
Nottingham & Wien

Thanks for your attention :)

Got interested?

Contact me!

hi@tobi-haas.de



Backup

Classical Information

- A single coin

Events Blue Red

Mixture $p_1\mathrm{Blue} + p_2\mathrm{Red}$

- Entropy = Mean missing information

$S(p) = -\sum p_i \ln p_i$

  - Increases with unlikelyness

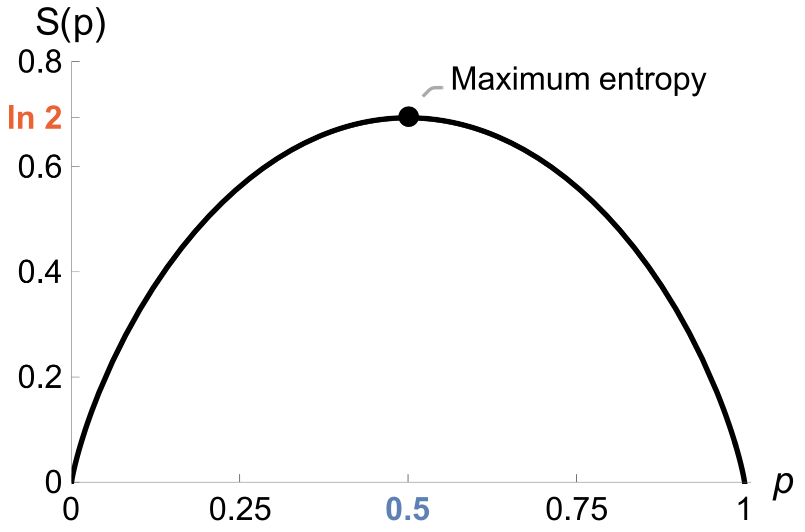
$S(p) \sim p_i^{-1}$

  - Information of independent systems adds up

$p_A \times p_B \to S(p_A) + S(p_B)$

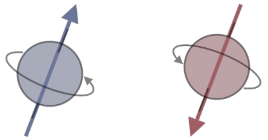
  - Non-negative (zero iff outcome is known)

$S(p) \geq 0, \quad S(p) = 0 \Leftrightarrow p_i = 1 \text{ for one } i$



Quantum information

- A single QBit



Pure states $|1\rangle$ $|0\rangle \in \mathcal{H}$ Mixed states $\boldsymbol{\rho} = p|\mathbf{1}\rangle\langle\mathbf{1}| + (1-p)|\mathbf{0}\rangle\langle\mathbf{0}|$

- Quantum entropy = Mean missing information $S(\boldsymbol{\rho}) = -\mathbf{Tr}\{\boldsymbol{\rho} \ln \boldsymbol{\rho}\}$
  - Increases with mixedness

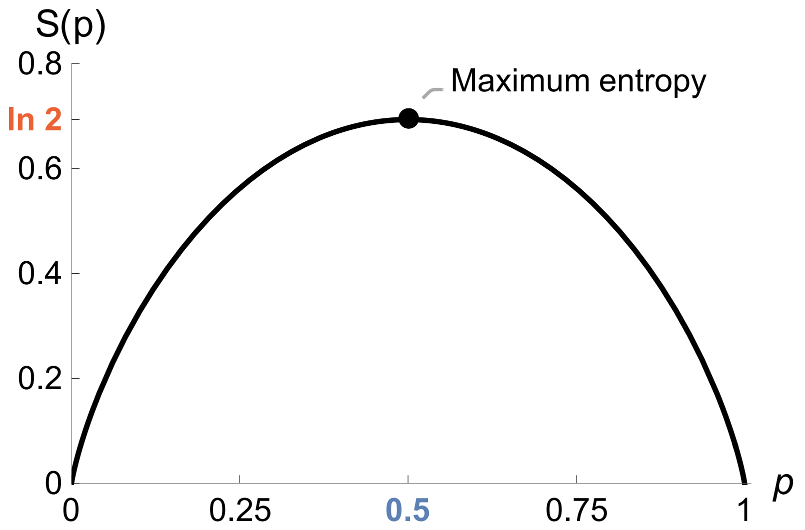
    $S(\boldsymbol{\rho}) \sim \boldsymbol{\rho}^{-1}$

  - Information of independent systems adds up

    $\boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B \rightarrow S(\boldsymbol{\rho}_A) + S(\boldsymbol{\rho}_B)$

  - Non-negative (zero iff state is pure)

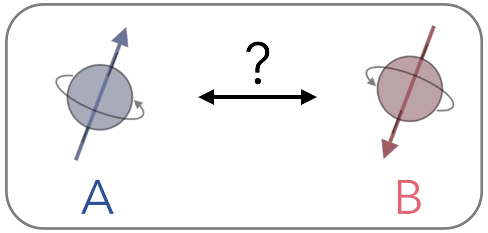
    $S(\boldsymbol{\rho}) \geq 0, \;\; S(\boldsymbol{\rho}) = 0 \Leftrightarrow \boldsymbol{\rho} = |\psi\rangle\langle\psi|$



Entropy & entanglement

- Two QBits

Global $\boldsymbol{\rho}$



$$S(\boldsymbol{\rho}) = -\mathbf{Tr}\{\boldsymbol{\rho} \ln \boldsymbol{\rho}\}$$

Local $\boldsymbol{\rho}_A = \mathbf{Tr}_B\{\boldsymbol{\rho}\}$ $\boldsymbol{\rho}_B = \mathbf{Tr}_A\{\boldsymbol{\rho}\}$

- Classical correlations (separable states, e.g. $\boldsymbol{\rho} = \boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B$)

$$S(\boldsymbol{\rho}) \geq S(\boldsymbol{\rho}_A), S(\boldsymbol{\rho}_B) \quad \rightarrow \quad S(\boldsymbol{\rho}) = 0 \Rightarrow S(\boldsymbol{\rho}_A), S(\boldsymbol{\rho}_B) = 0$$

→ *Know the system and **thus** know **all** about its parts*

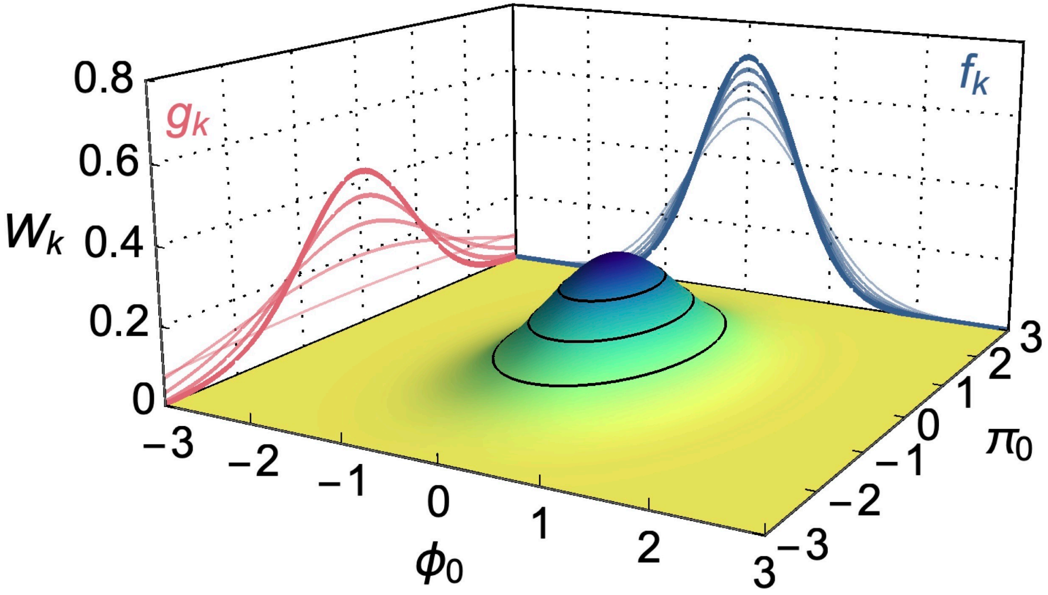
- Entanglement

$$S(\boldsymbol{\rho}) < S(\boldsymbol{\rho}_A), S(\boldsymbol{\rho}_B) \quad \rightarrow \quad S(\boldsymbol{\rho}) = 0 \quad \wedge \quad S(\boldsymbol{\rho}_A), S(\boldsymbol{\rho}_B) > 0$$

→ *Know the system and may know **nothing** about its parts*

Particles in phase space

- Wigner $\mathcal{W}$-distribution for free particle

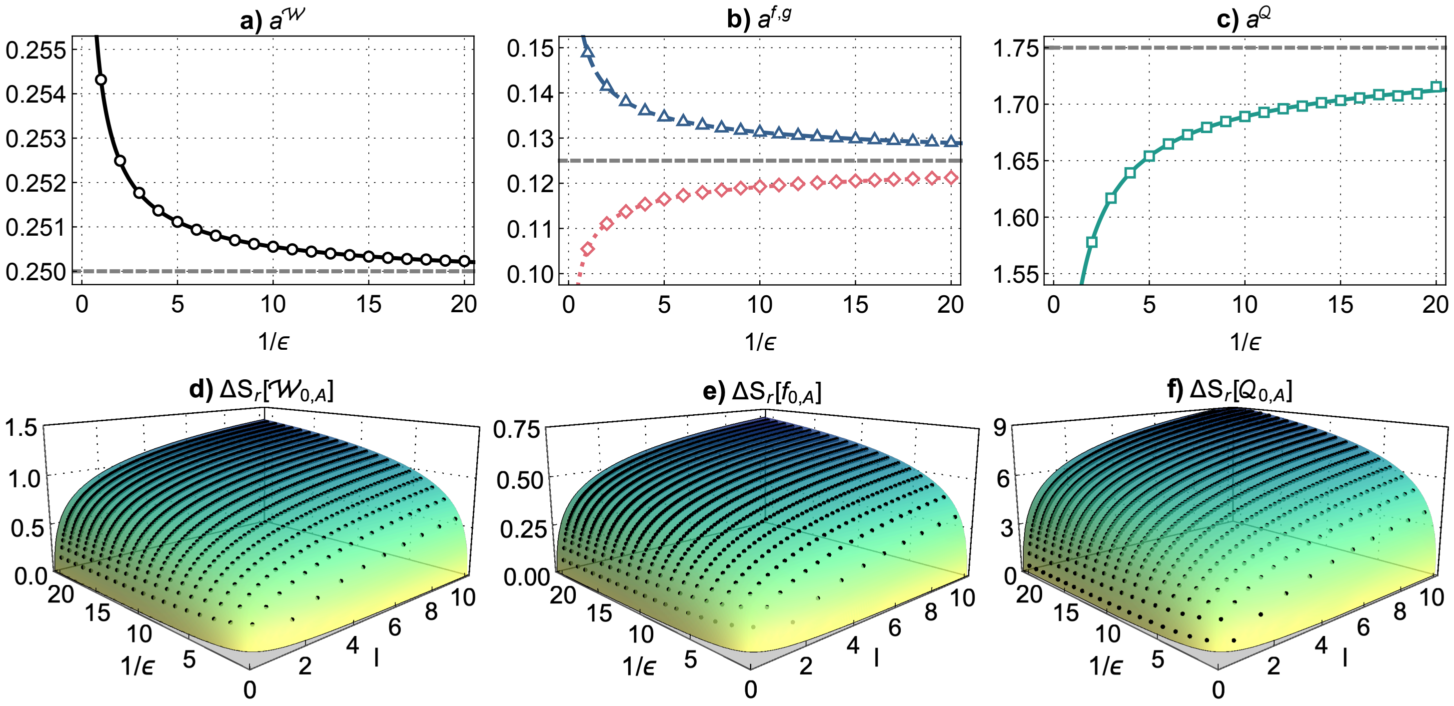


Central charge

Subtracted classical entropies | **Scalar field** | BEC | Area law → Volume law | **Backup**

- Ground state entropies

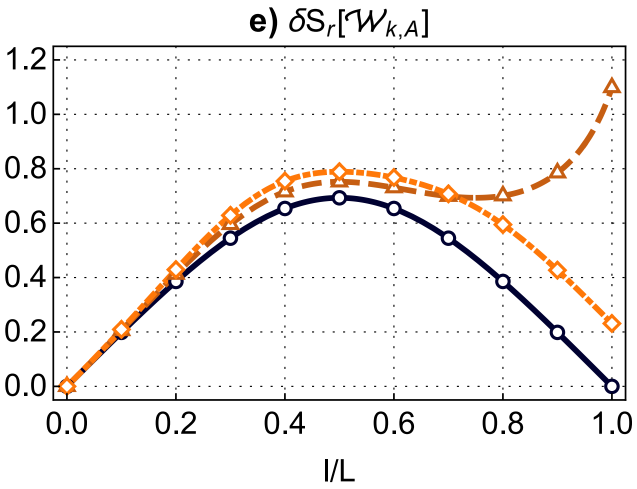
$$\Delta S_r[\mathcal{W}_{0,A}] = \Delta S_r[f_{0,A}] + \Delta S_r[g_{0,A}] = \frac{c}{4}\ln\frac{l}{\epsilon}$$



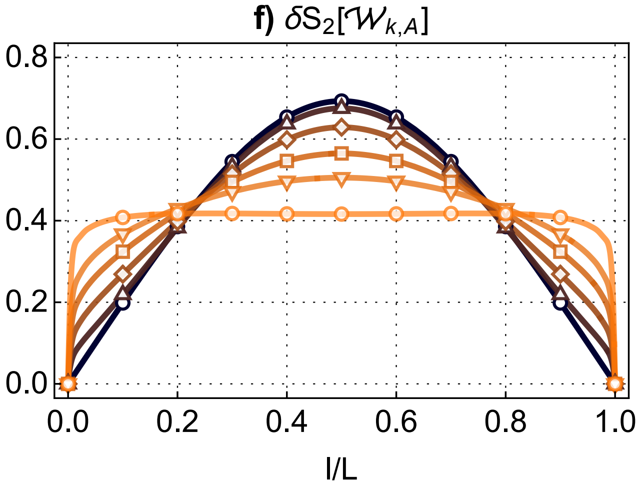
Particles – Other quantities I

- High particle energies: $\omega(p) = \sqrt{m^2 + p^2} \gg \frac{1}{l}, \frac{1}{L-l}$

$$\delta S_r\left[\mathcal{O}_{k,A}\right] = \frac{1}{1-r} \ln\left[1 + \sum_{i=1}^{r} a_{r,i} \left(\frac{l}{L}\right)^i\right]$$



$$\delta S_r\left[\mathcal{O}_{k,A}\right] \sim (2)\frac{l}{L}$$
$$\sim \delta S_2\left[\boldsymbol{\rho}_{k,A}\right]$$



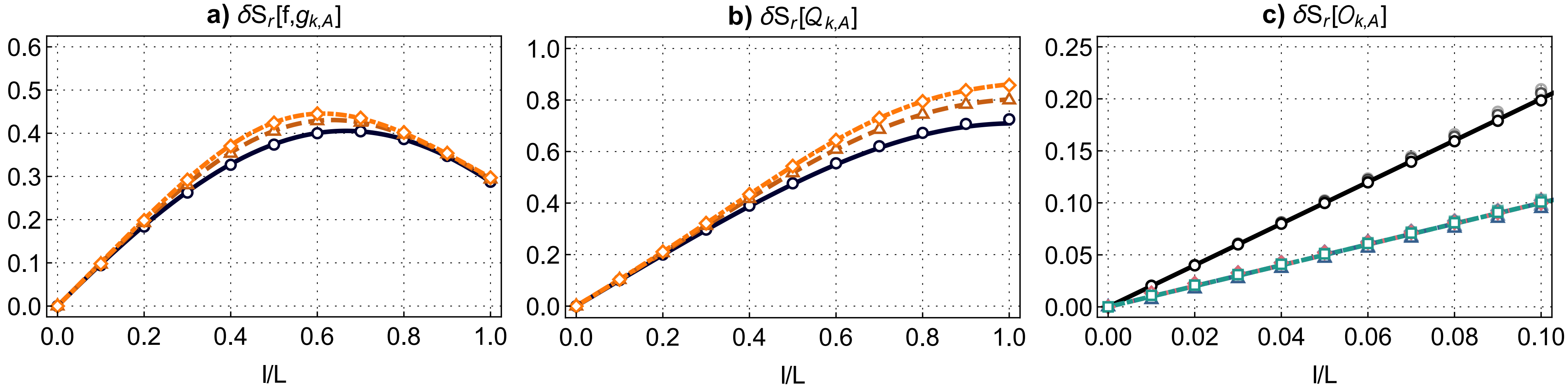
small energies

$$\delta S_2\left[\mathcal{W}_{k,A}\right] \rightarrow const.$$

Particles – Other quantities II

- High particle energies: $\omega(p) = \sqrt{m^2 + p^2} \gg \frac{1}{l}, \frac{1}{L-l}$

$$\delta S_r\left[\mathcal{O}_{k,A}\right] = \frac{1}{1-r} \ln\left[1 + \sum_{i=1}^{r} a_{r,i} \left(\frac{l}{L}\right)^i\right]$$



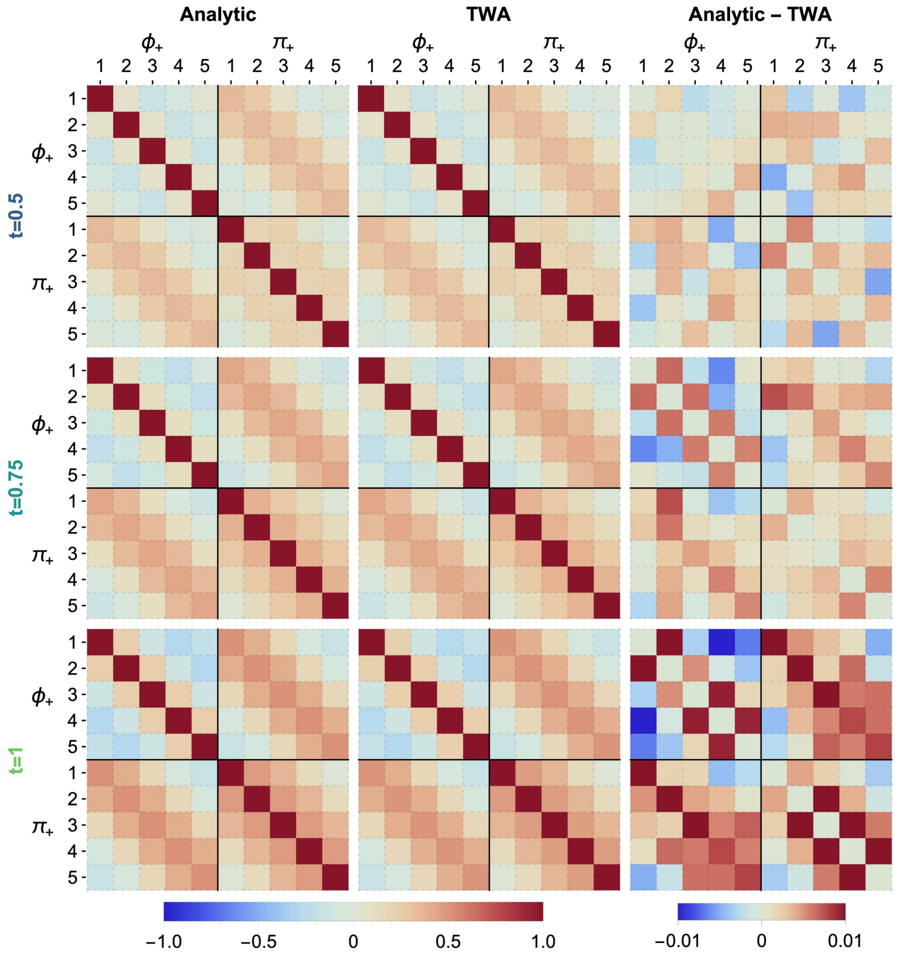
Parameters

- Experimental values

Total system size	$N = 20$
Subsystem size	$M = 5$
Energy scale	$c_1 = -1/n$
Number of atoms	$n = 10^3$ per well
Spin coupling	$c_0 = -2c_1$ (^{7}Li)
2nd order Zeemann shift	$q = 2J$
Coupling between wells	$J = 2$
Samples	$N_s = 10^4$

Correlation matrices: Gaussian vs. TWA

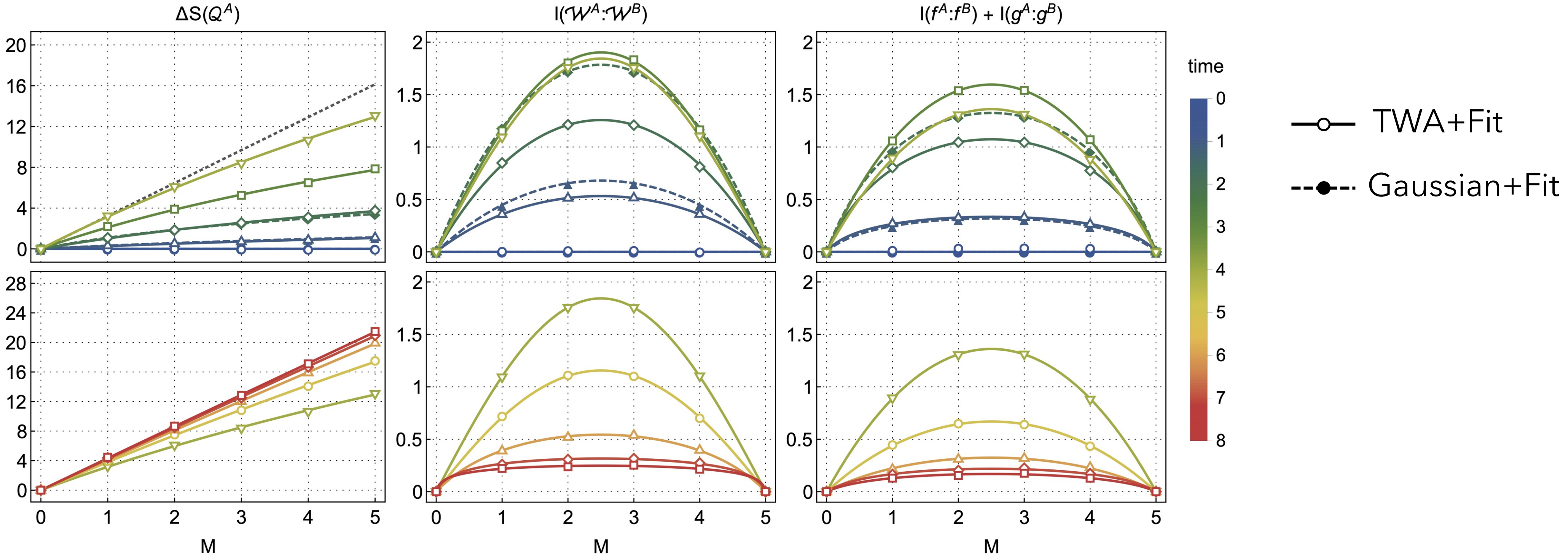
Subtracted classical entropies | Scalar field | **BEC** | Area law → Volume law | **Backup**



Time evolution - Other quantities

Subtracted classical entropies | Scalar field | BEC | Area law → Volume law | Backup

- Early times + late times

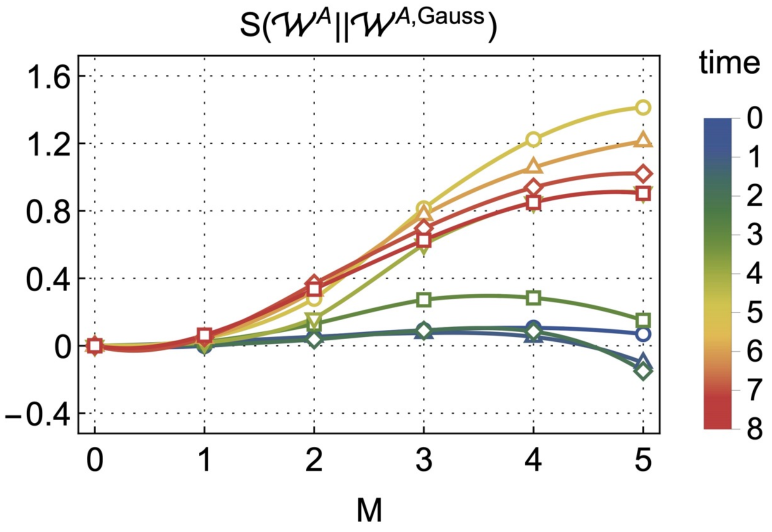


Non-Gaussianity

Subtracted classical entropies | Scalar field | BEC | Area law → Volume law | Backup

- Relative Wigner entropy

$$S\left[\mathcal{W}^A \| \mathcal{W}^{A,\mathrm{Gauss}}\right] = \int \mathcal{D}v_A\, \mathcal{W}^A \left(\ln \mathcal{W}^A - \ln \mathcal{W}^{A,\mathrm{Gauss}}\right)$$

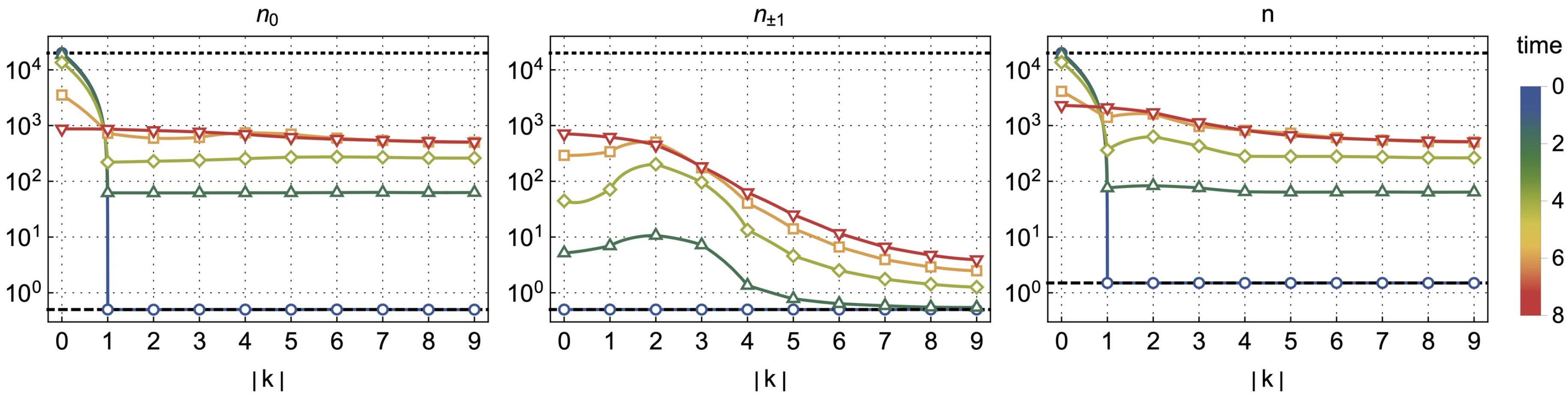


→ Non-Gaussian features in higher dimensions

Mode occpuations

Subtracted classical entropies | Scalar field | BEC | Area law → Volume law | Backup

- Populations of momentum modes



→ Mesoscopic occupations justify TWA for late times