Centre for Quantum Technologies

NUS National University of Singapore

Simulating Entanglement with 1 bit of Communication

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YQIS 2024

Outline

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1. The problem

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1. The problem
2. ML approach

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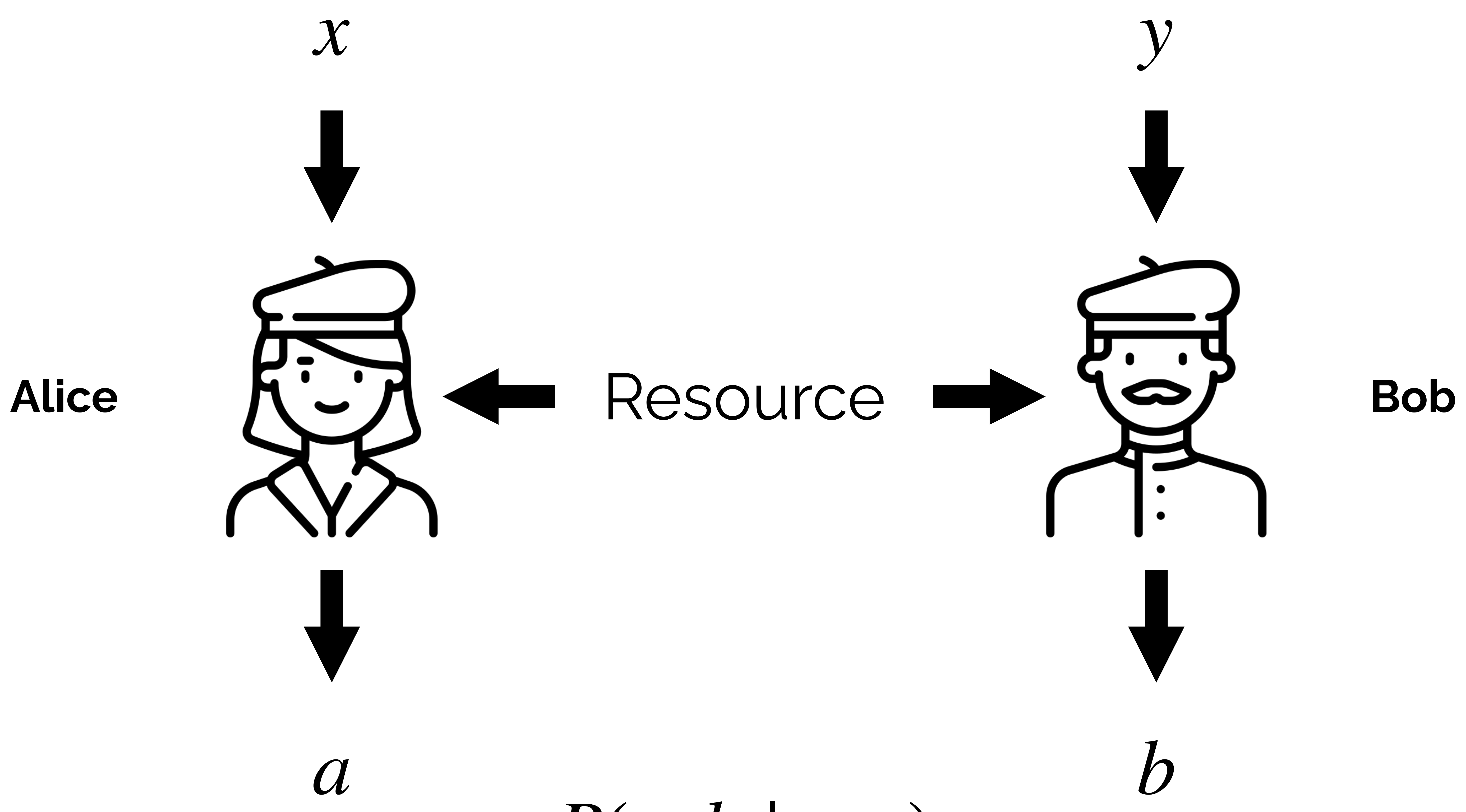
2. ML approach
3. Bell inequalities for 1-bit of communication

The problem

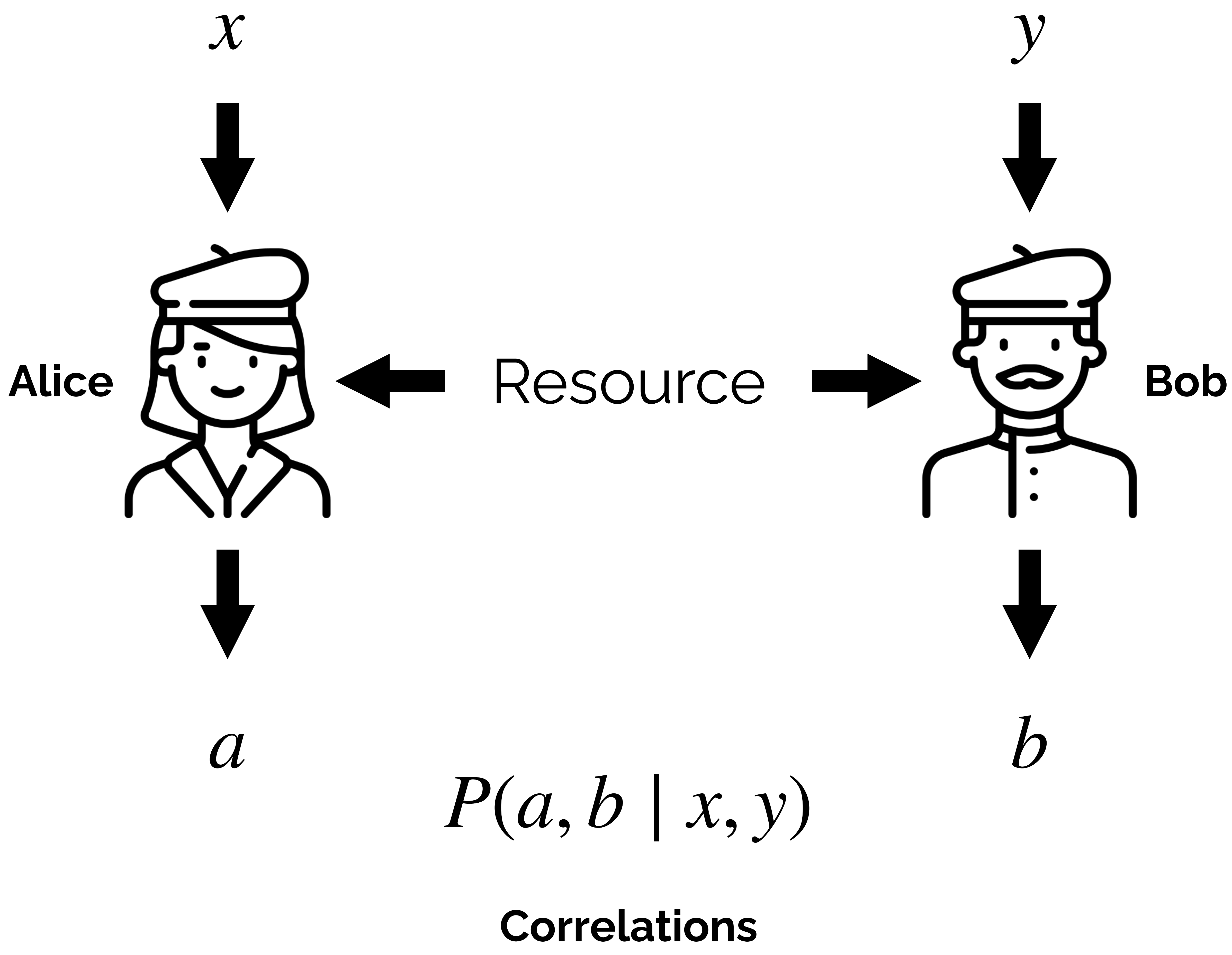
Bell's theorem



$$P(a, b \mid x, y)$$

Correlations

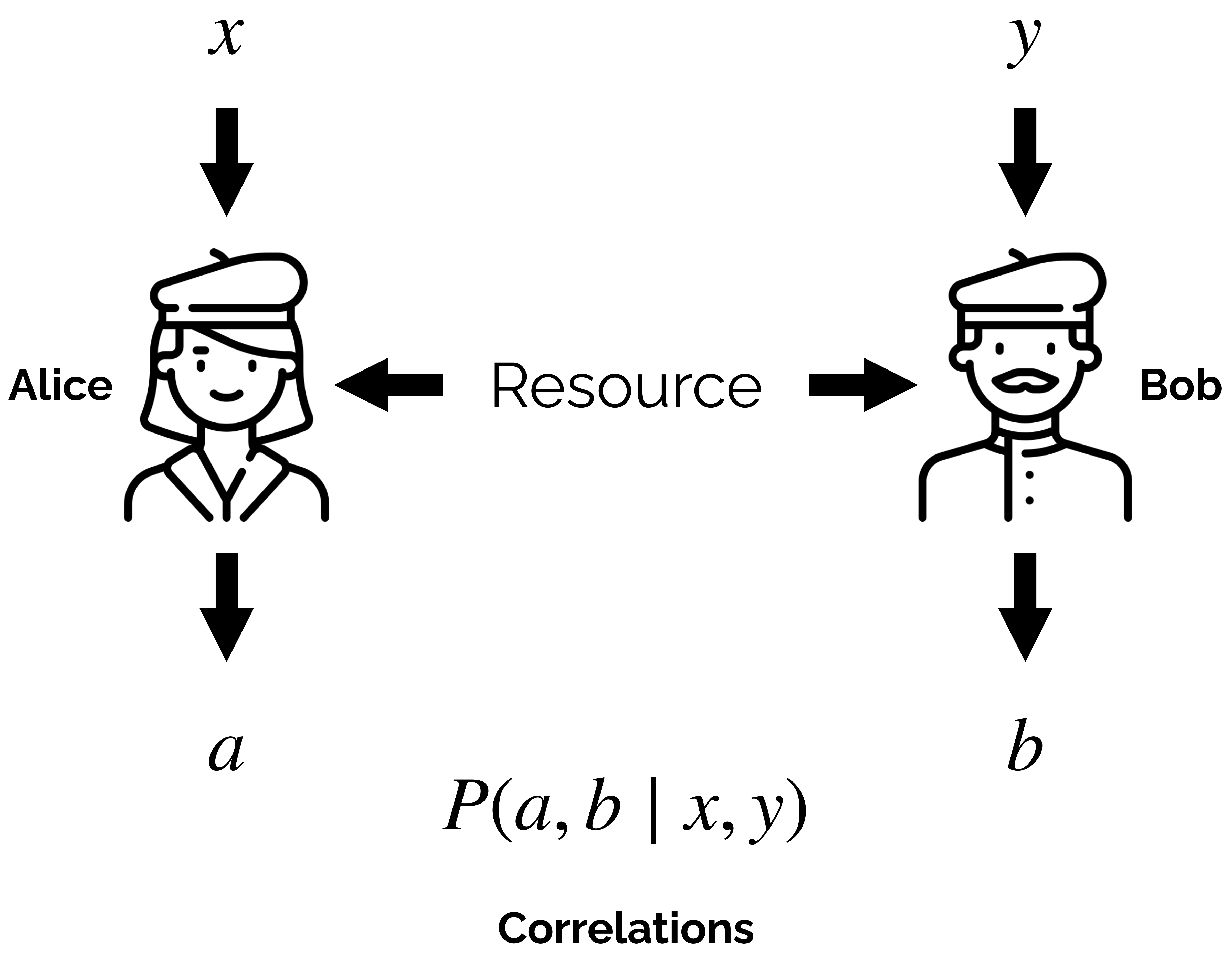
Bell's theorem



Bell's theorem

- No-signalling
  - Alice cannot know Bob's input just from her output

$$P(a \mid x, y) = P(a \mid x, y') = P(a \mid x) \quad \forall a, x, y, y'$$



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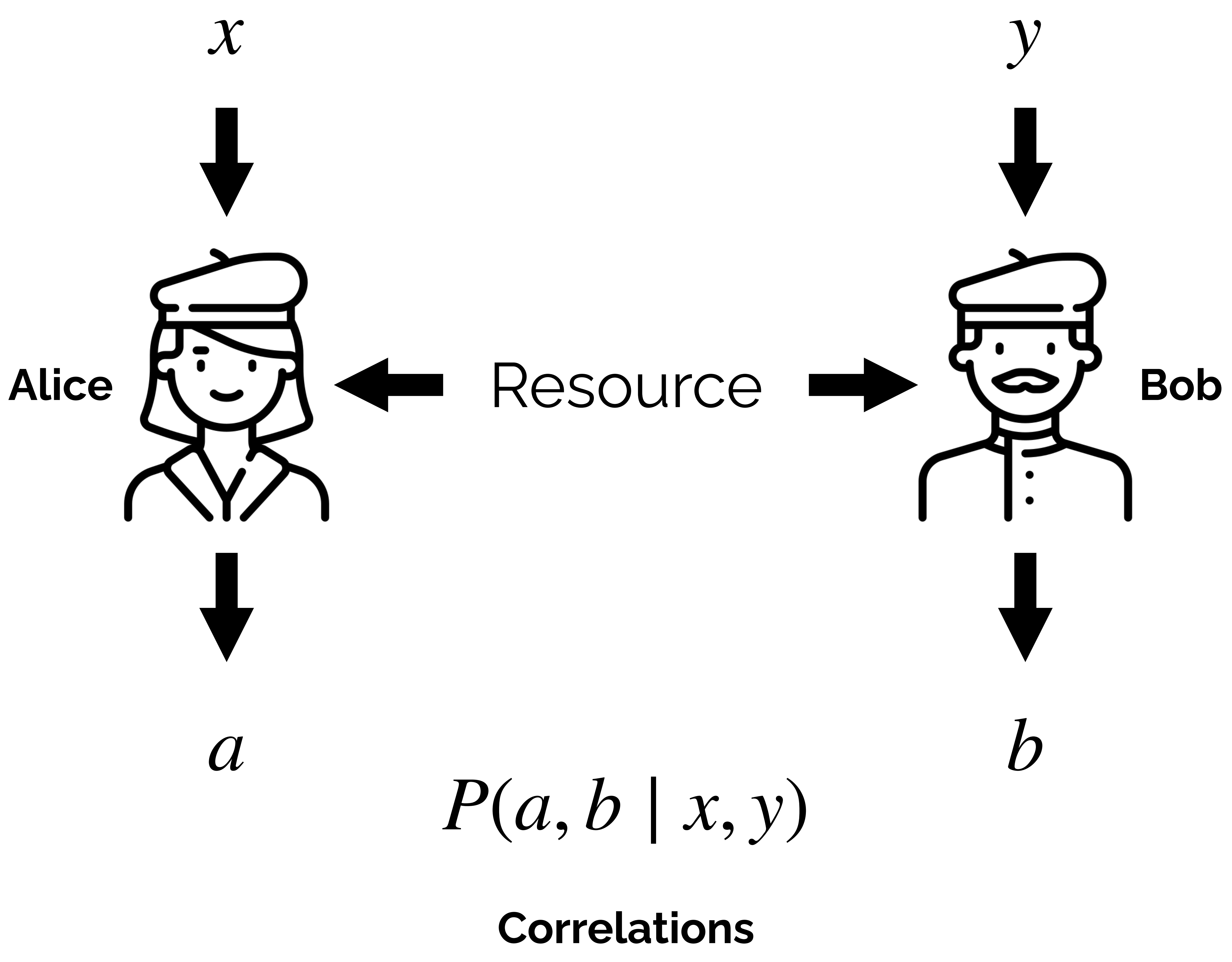
- LHV
  - Their output are factorisable after knowing λ

$$P(a, b \mid x, y) = \int d\lambda\, q(\lambda)\, P(a \mid x, \lambda)\, P(b \mid y, \lambda)$$



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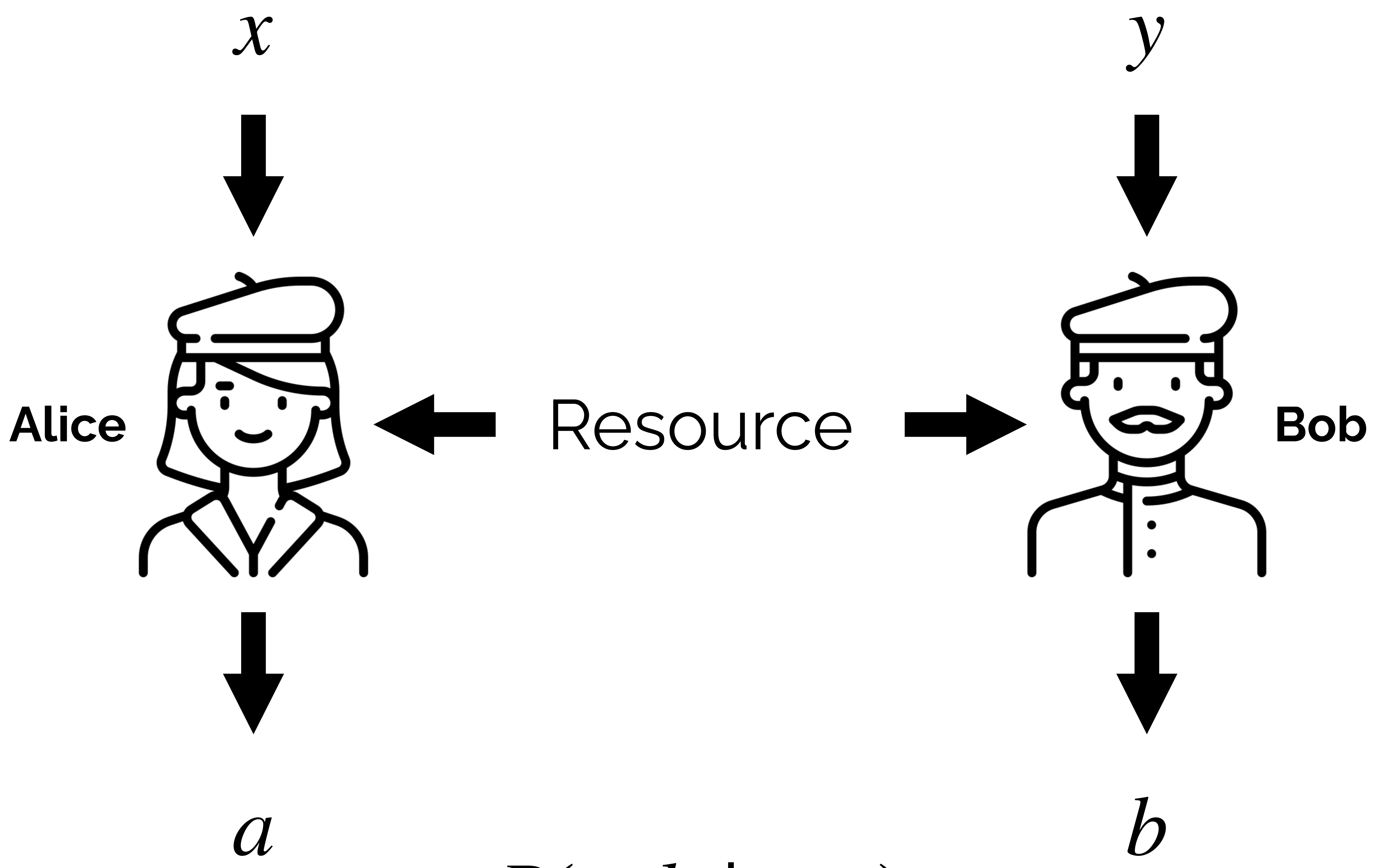

$$P(a, b \mid x, y)$$

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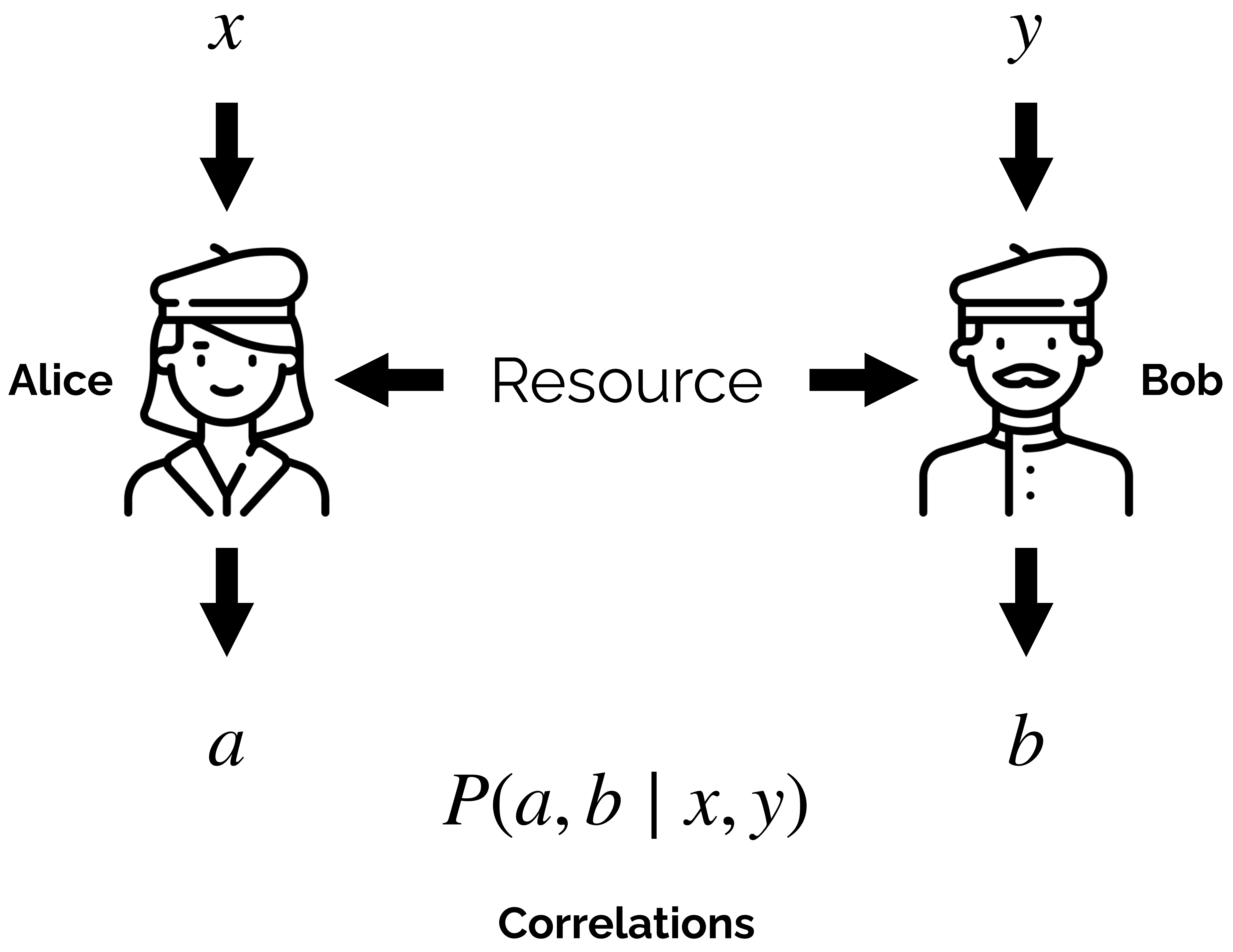
Bell's theorem

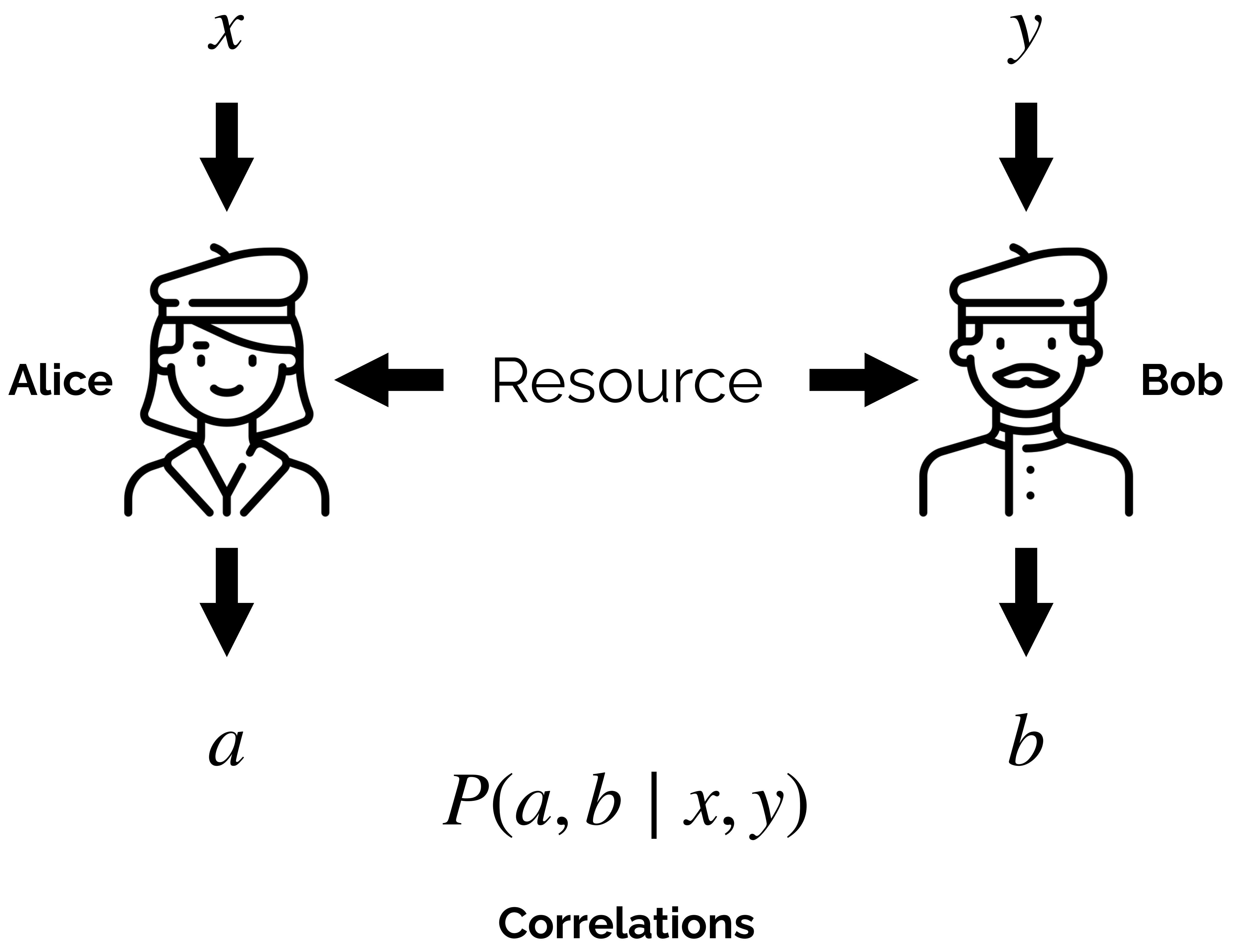
- Bell's theorem states that some quantum behaviours are not local



Bell's theorem

- Bell's theorem states that some quantum behaviours are not local
  - I.e., one cannot write for all inputs and outputs

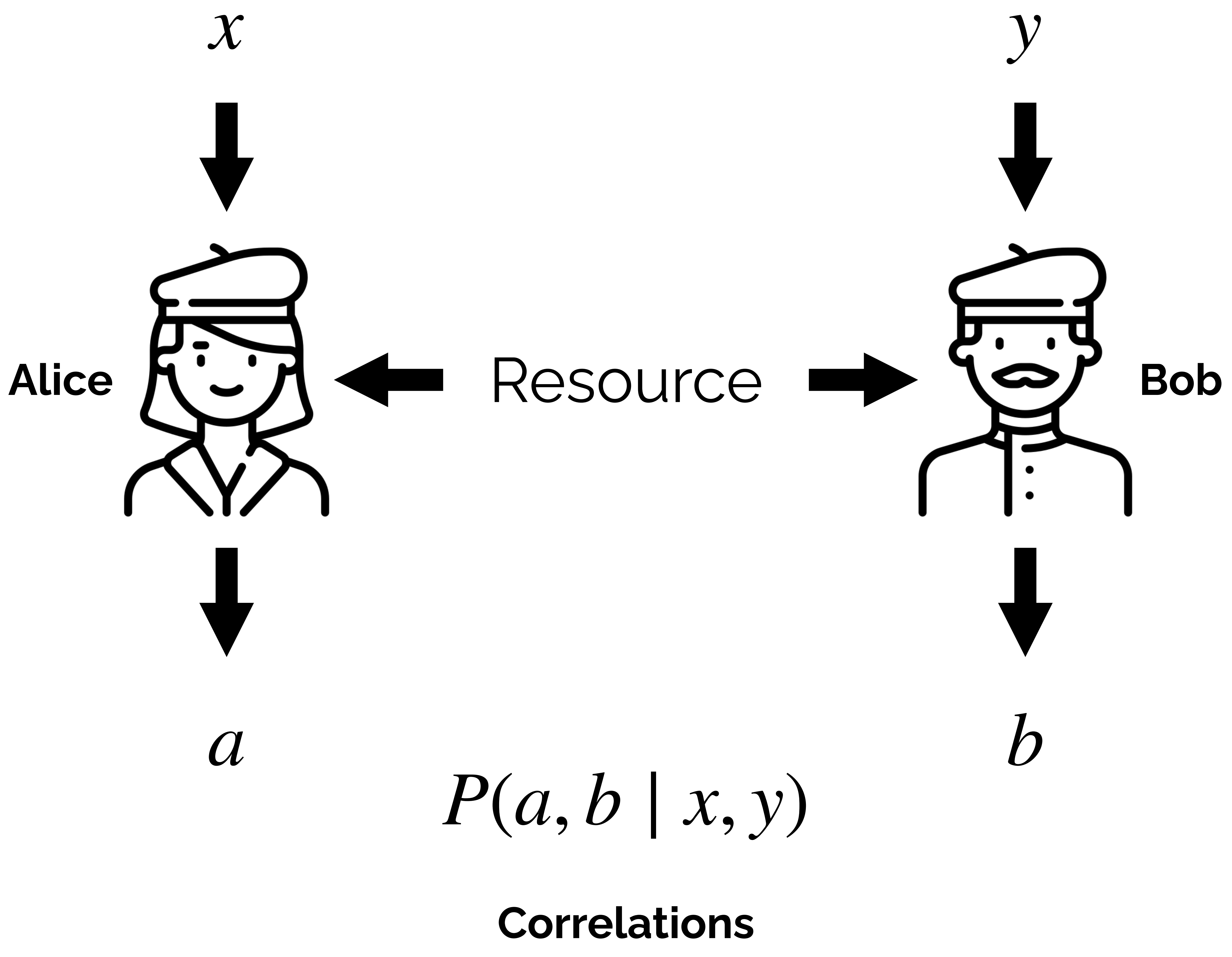
$$\mathrm{Tr}(\Pi_a^x \otimes \Pi_b^x \rho) \neq \int d\lambda\, q(\lambda)\, P(a \mid x, \lambda)\, P(b \mid y, \lambda)$$



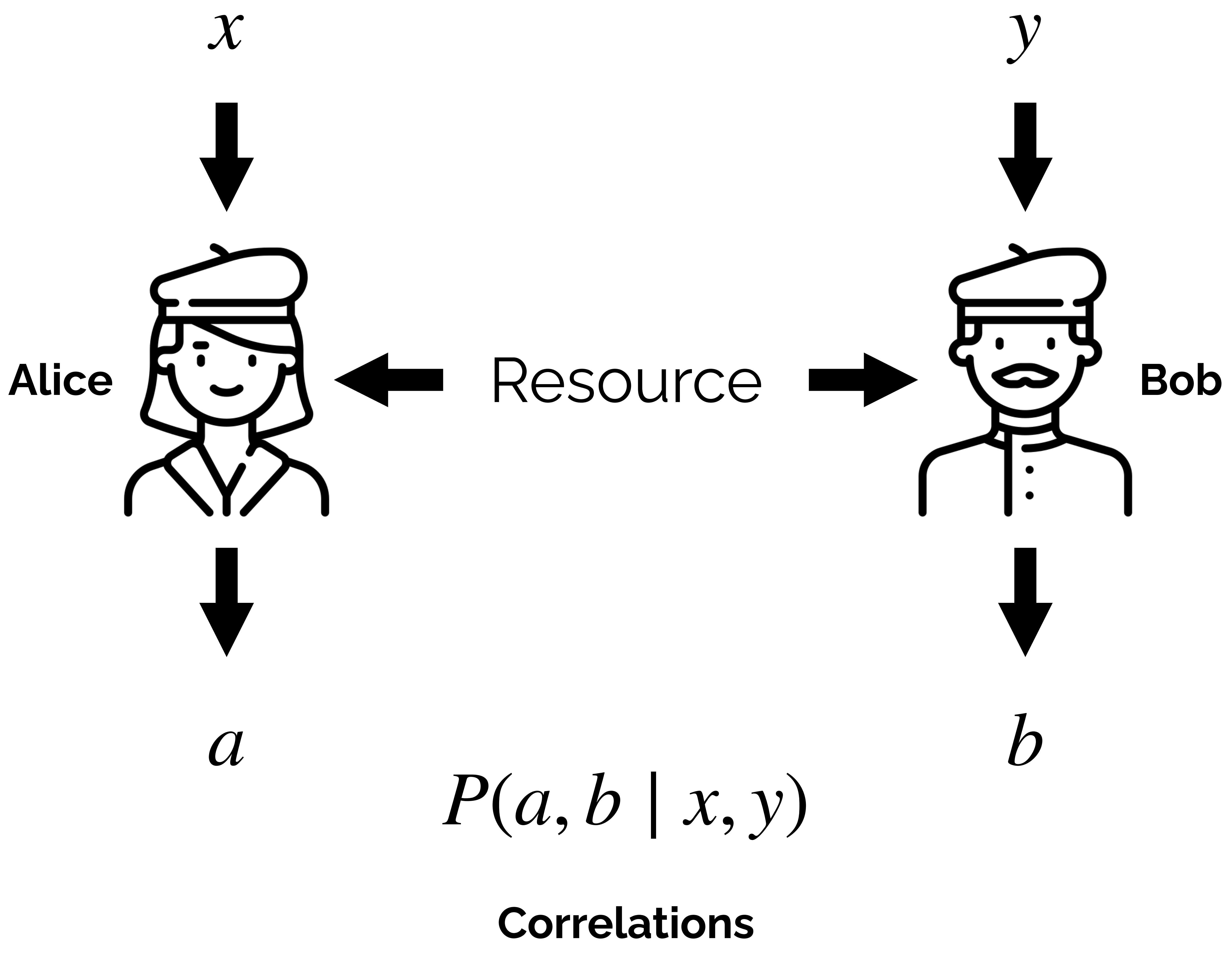
Bell's theorem

- Bell's theorem states that some quantum behaviours are not local
  - This is shown using a Bell inequality

$$\sum_{a,b,x,y} V_{a,b,x,y} P(a, b \mid x, y) \leq S_L$$

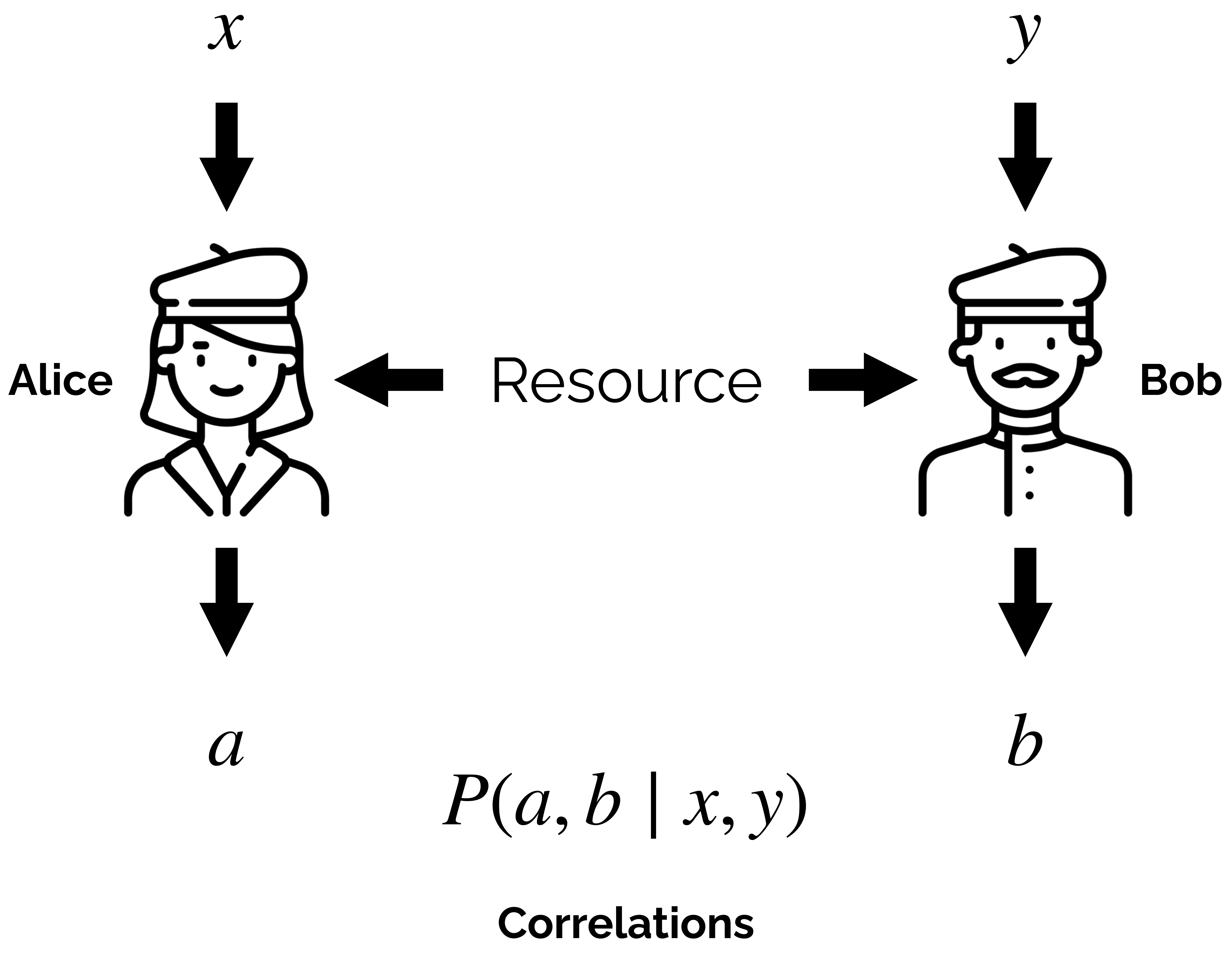


Bell's theorem



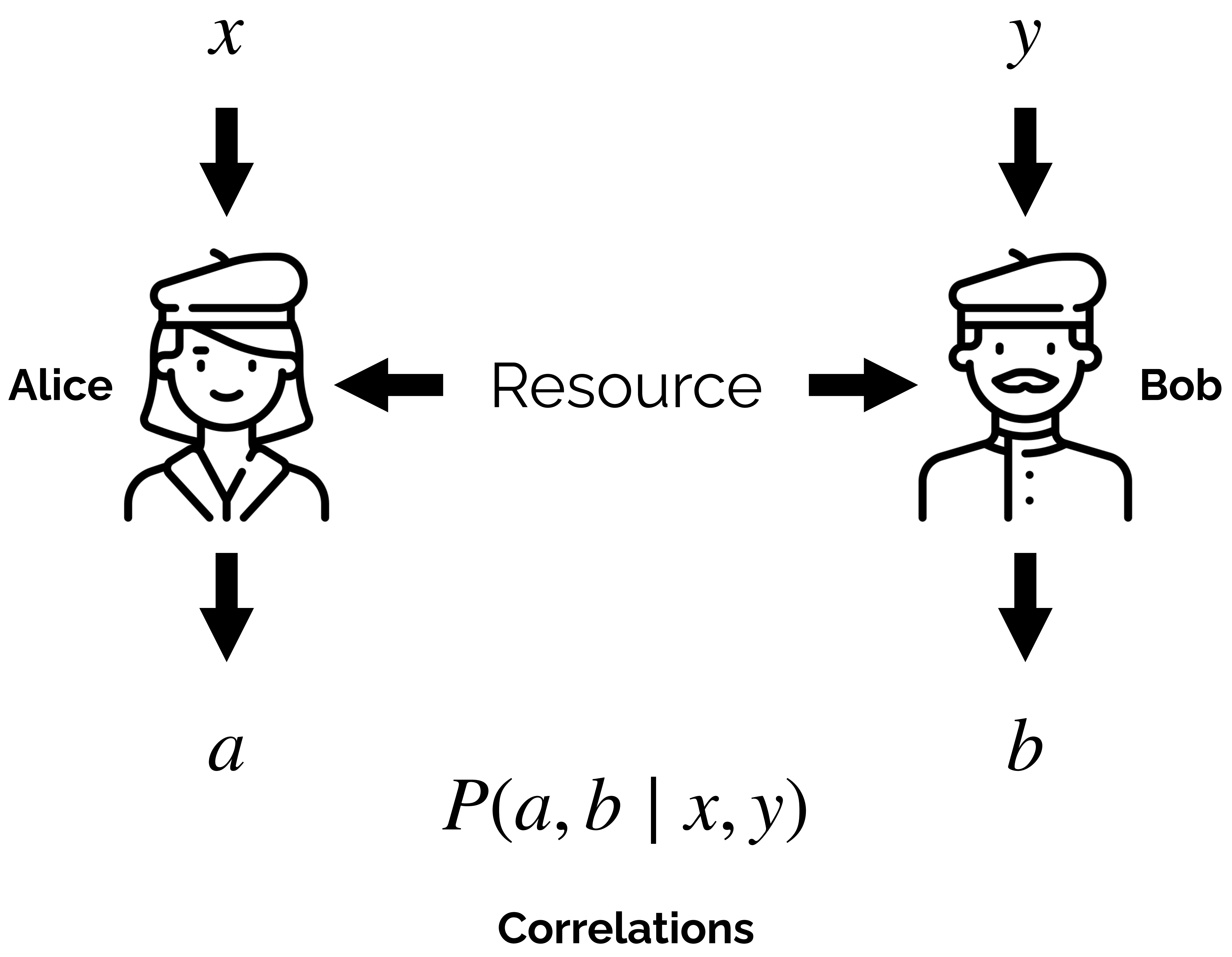
Bell's theorem

- There are many different inequalities, which are violated with different states to different degrees.

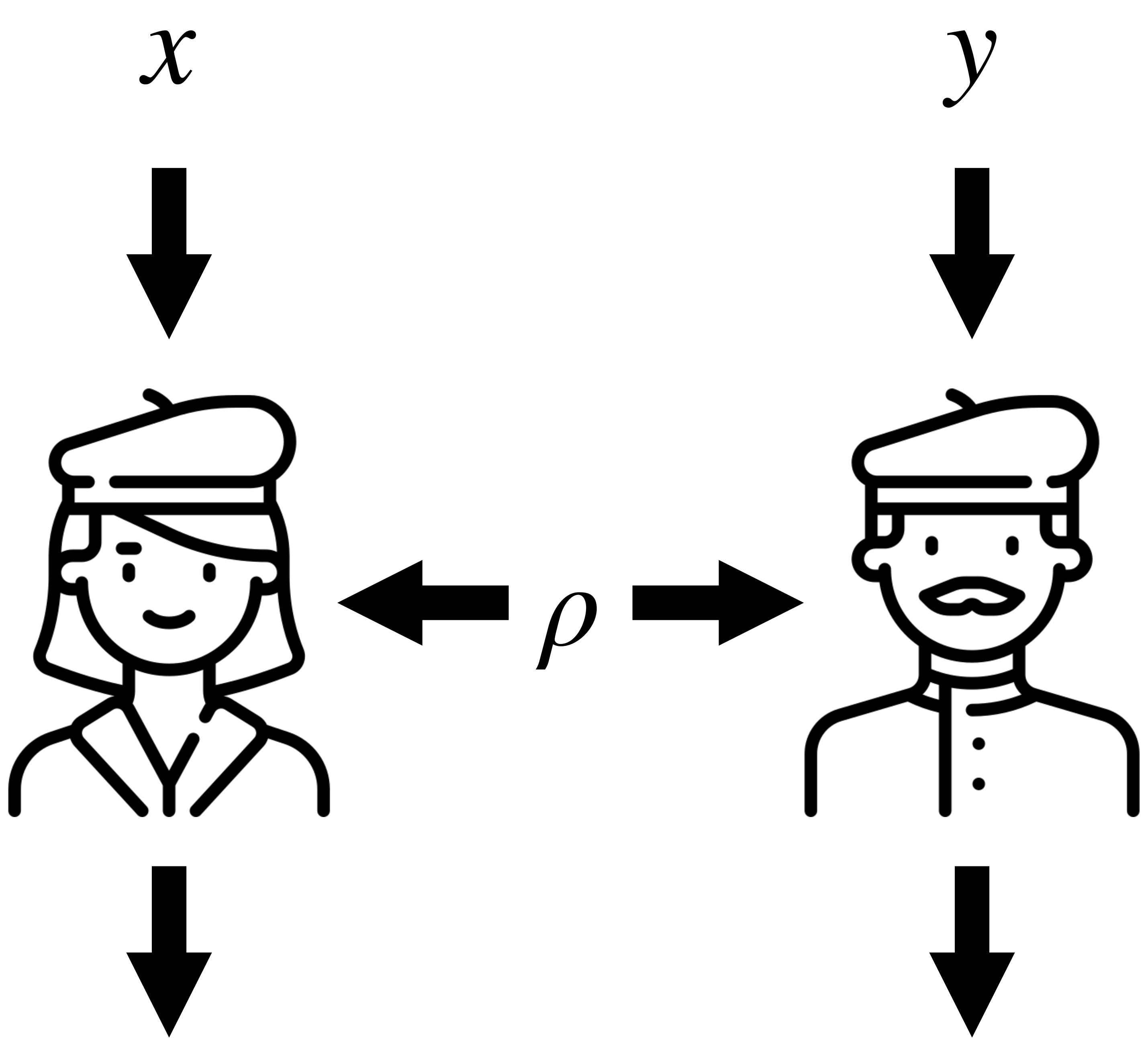


Bell's theorem

- There are many different inequalities, which are violated with different states to different degrees.
- Then, how do we quantify the 'strength' of a violation?



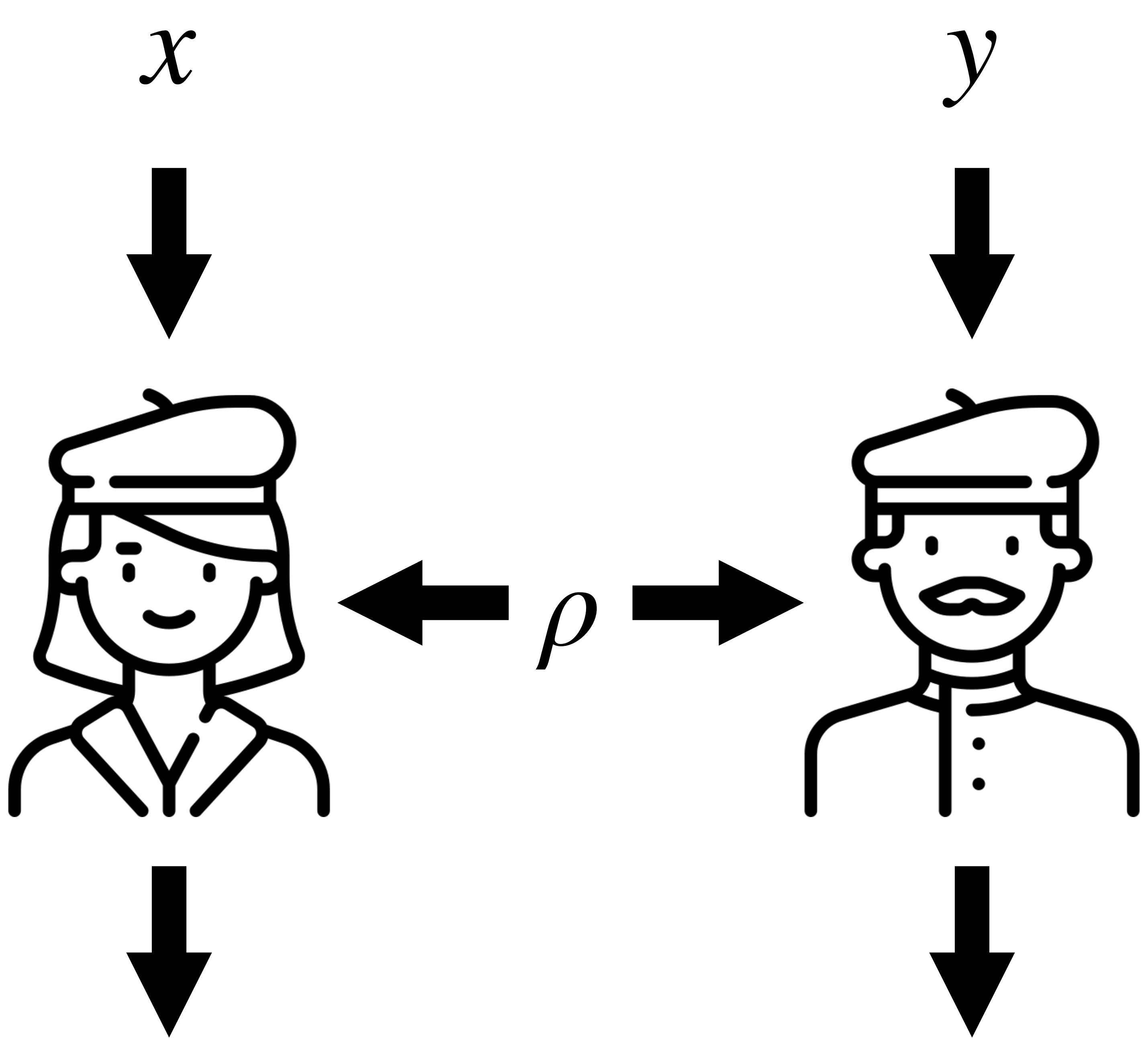
Entanglement simulation



$$P(a, b \mid x, y) = \mathrm{Tr}(\Pi_a^x \otimes \Pi_b^y \rho)$$

Quantum correlations

Entanglement simulation



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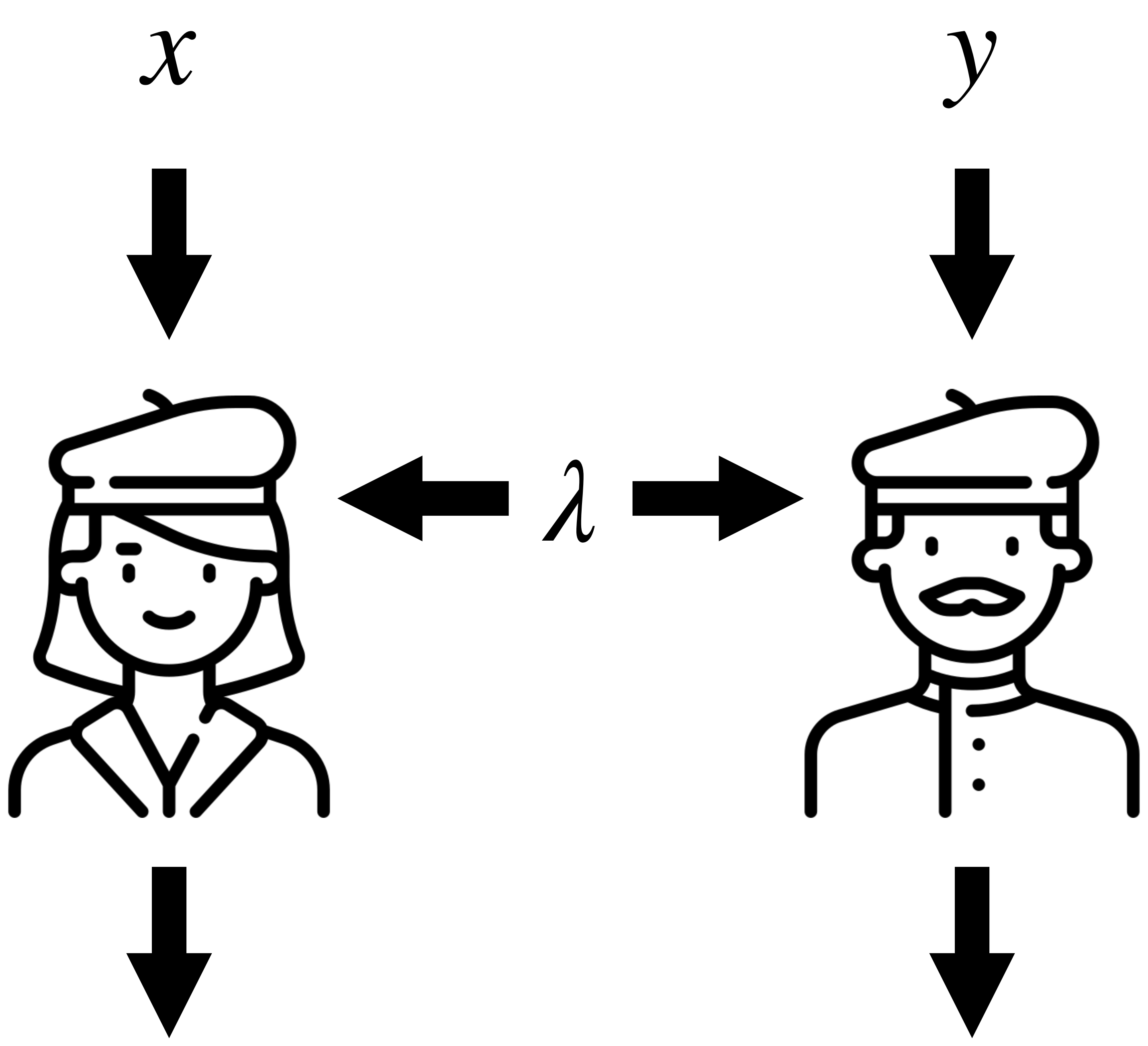
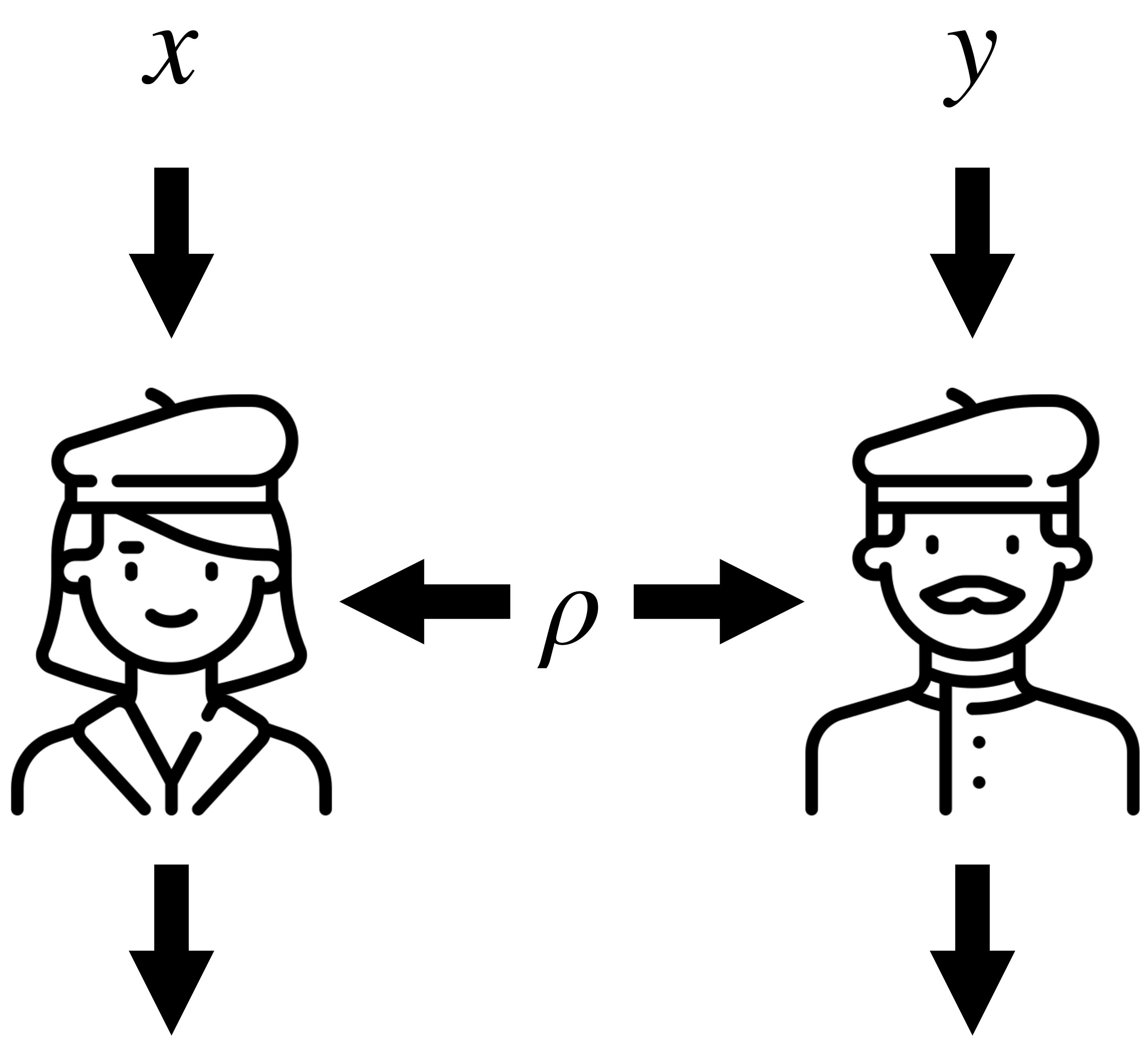


$$P(a, b \mid x, y) = \int d\lambda \, q(\lambda) \, P(a \mid x, \lambda) P(b \mid y, \lambda)$$

Local correlations

Entanglement simulation



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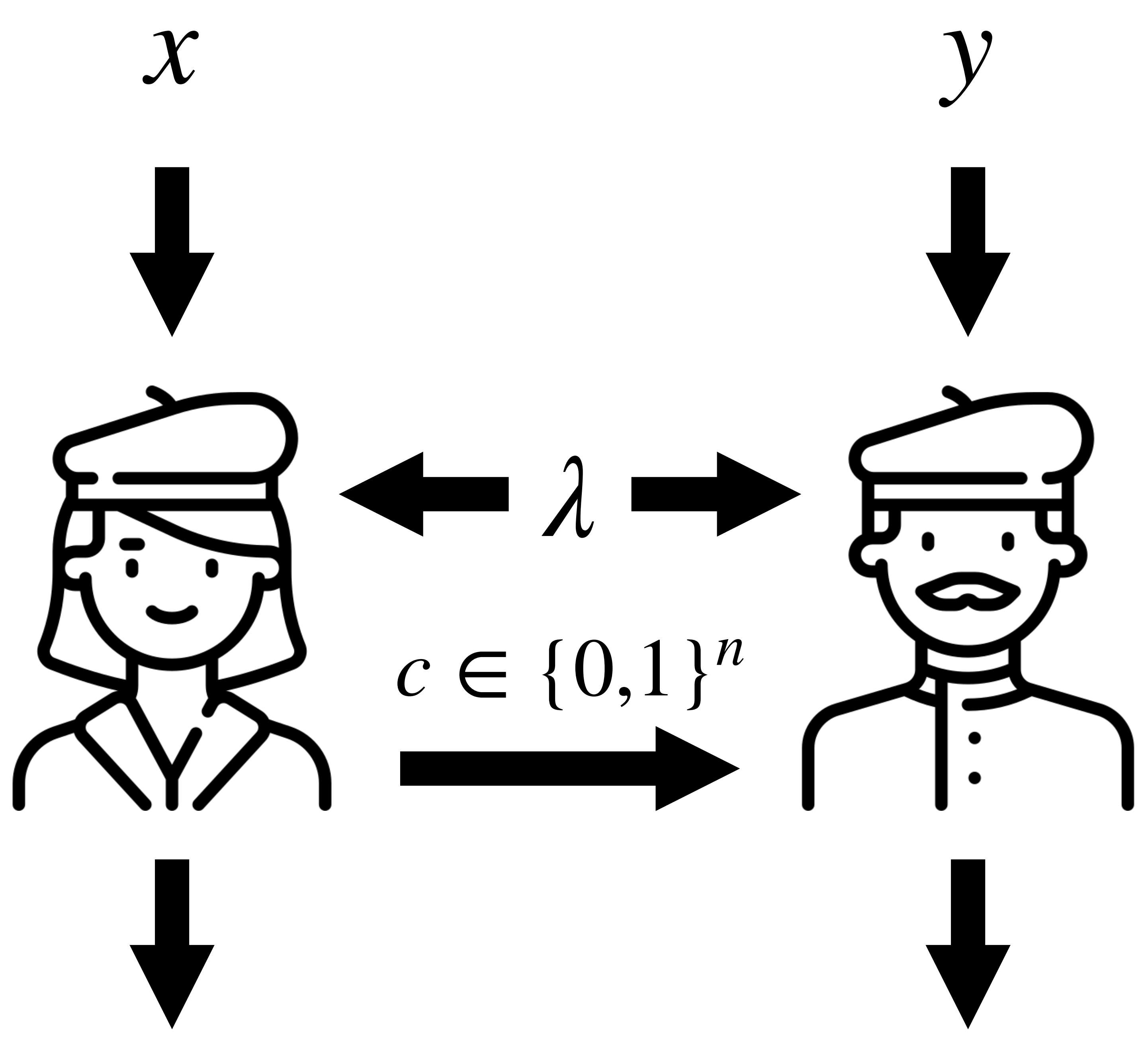

$$P(a, b \mid x, y) = \int d\lambda \, q(\lambda) \, P_{1-bit}(a, b \mid x, y, \lambda)$$

Local correlations with 1 bit of communication

Known results

Renner, M.J. and Quintino, M.T., 2023. The minimal communication cost for simulating entangled qubits. *Quantum*, *7*, p.1149.
Toner, B.F. and Bacon, D., 2003. Communication cost of simulating Bell correlations. Physical Review Letters, 91(18), p.187904. (figure)

Known results

- Consider the state

$$|\psi\rangle = \sqrt{d}\,|00\rangle + \sqrt{1-d}\,|11\rangle$$

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What about the rest of the two-qubit states?

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The Toner-Bacon protocol

$\hat{\lambda}_1, \hat{\lambda}_2$ are randomly uniformly distributed

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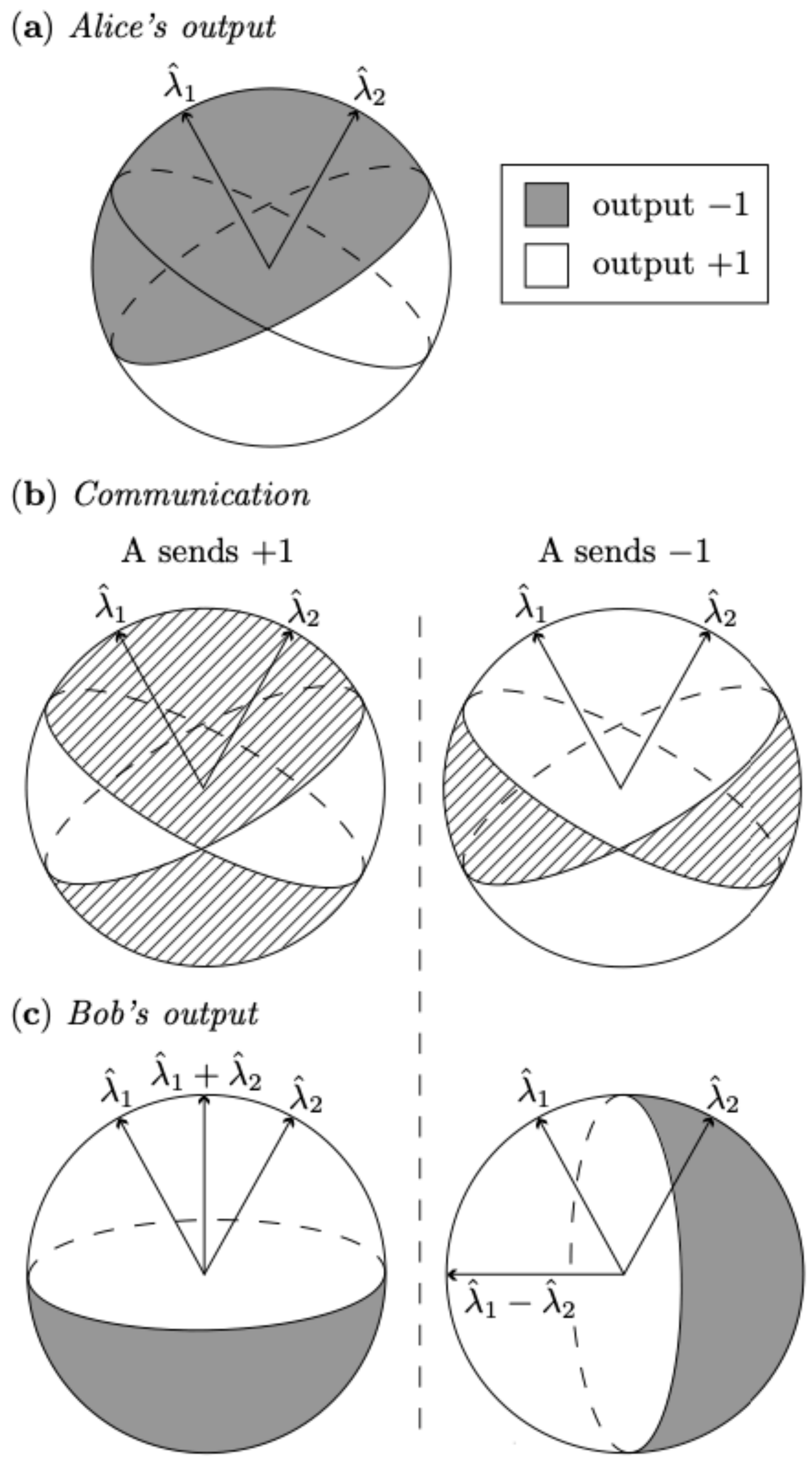

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Using ML to simulate entanglement

Quantum 7, 1150

Why use ML?

Krivàchy, T., Cai, Y., Cavalcanti, D., Tavakoli, A., Gisin, N. and Brunner, N., 2020. A neural network oracle for quantum nonlocality problems in networks. npj Quantum Information, 6(1), pp.1-7. (with image)

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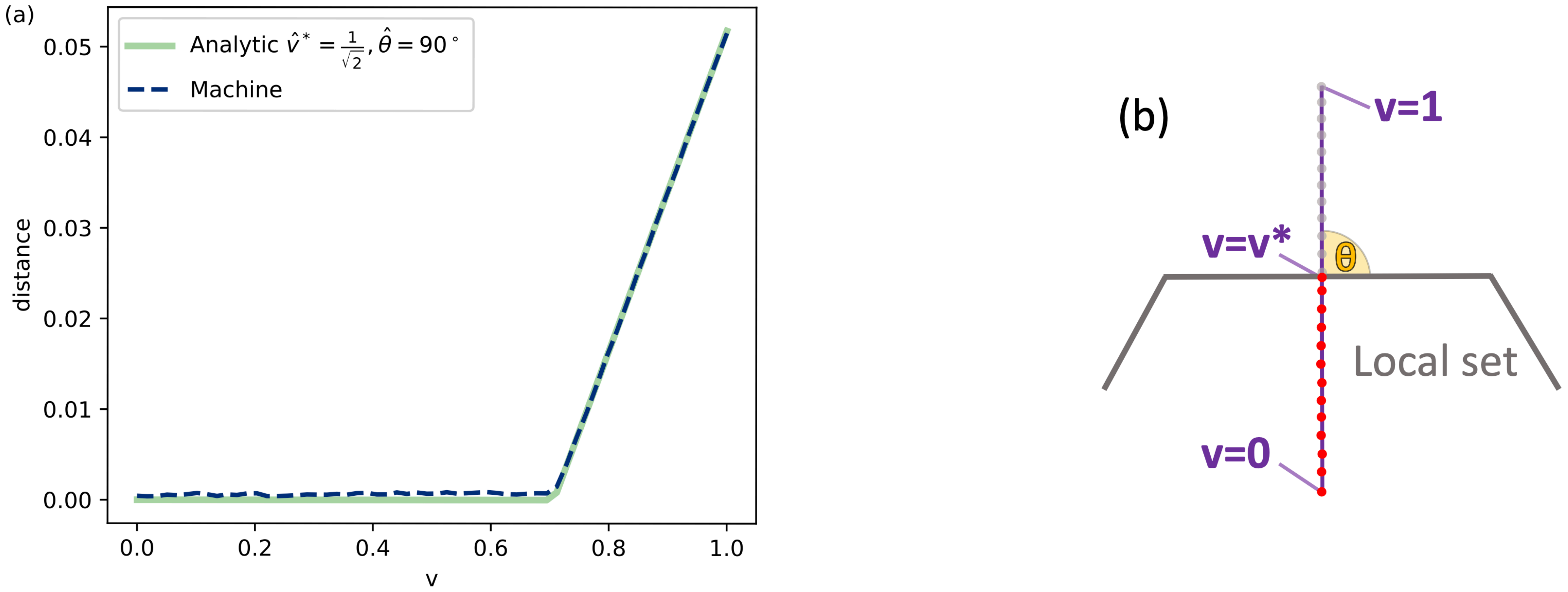

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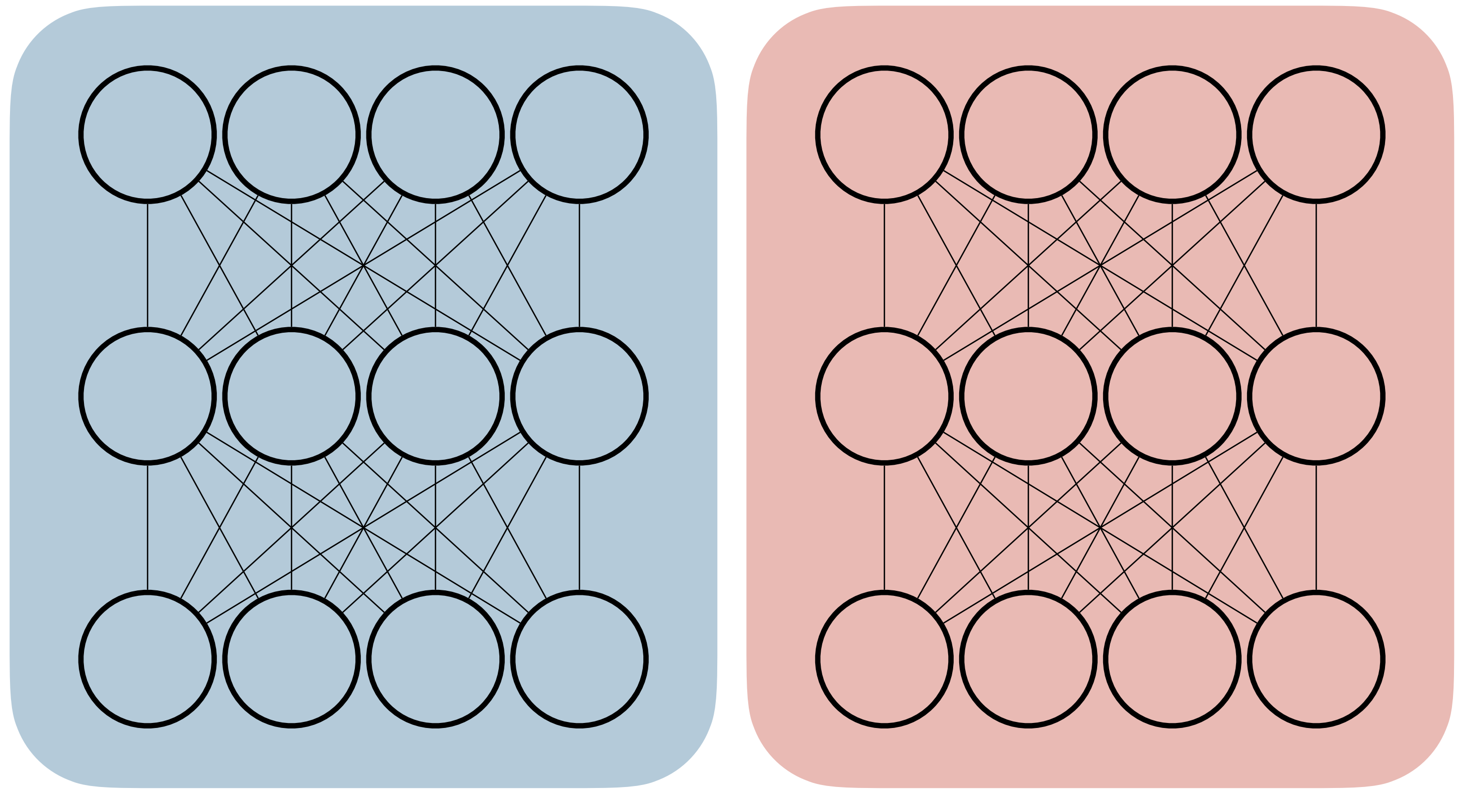
- The parties will be represented by a pair of NN.



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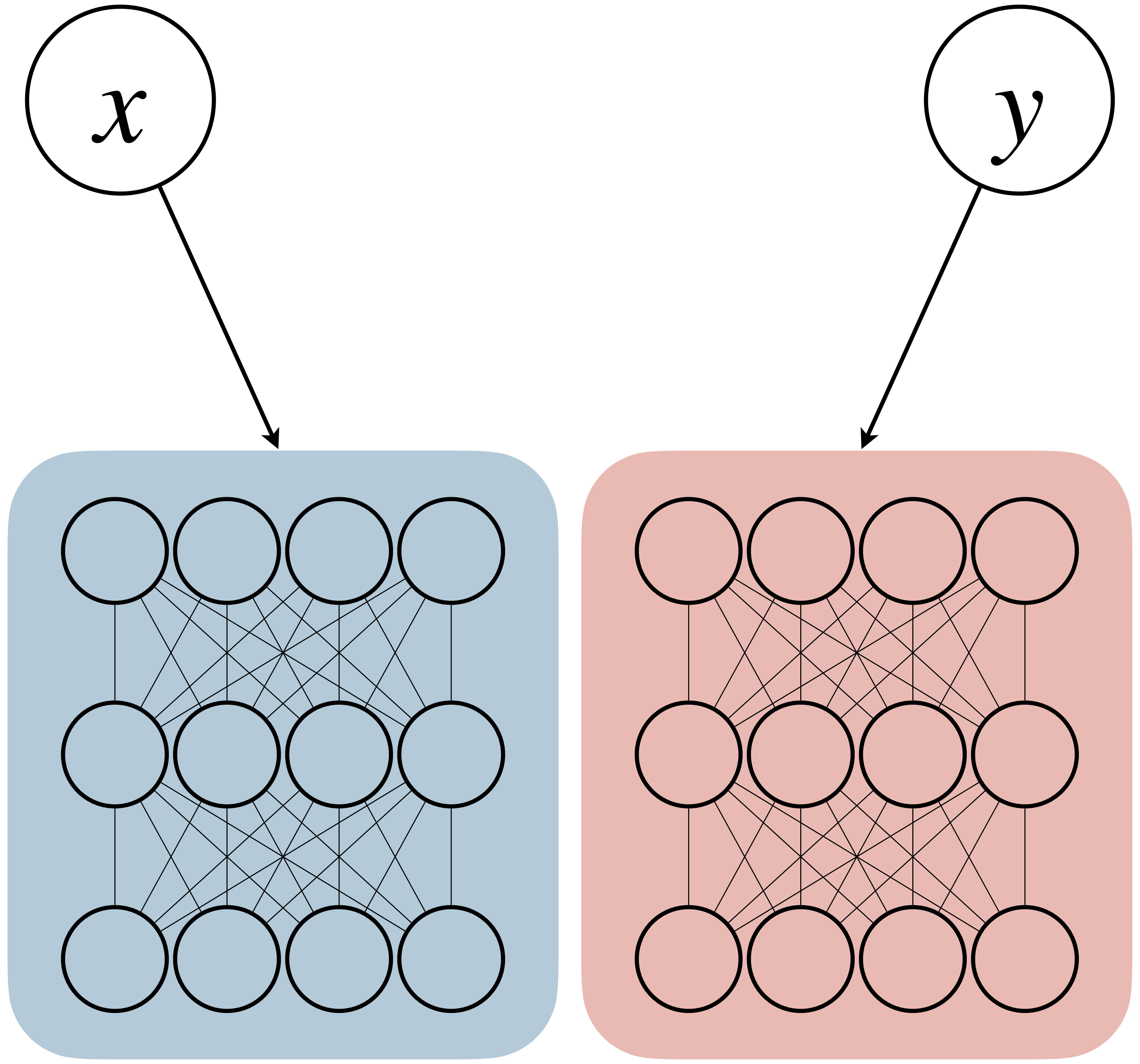

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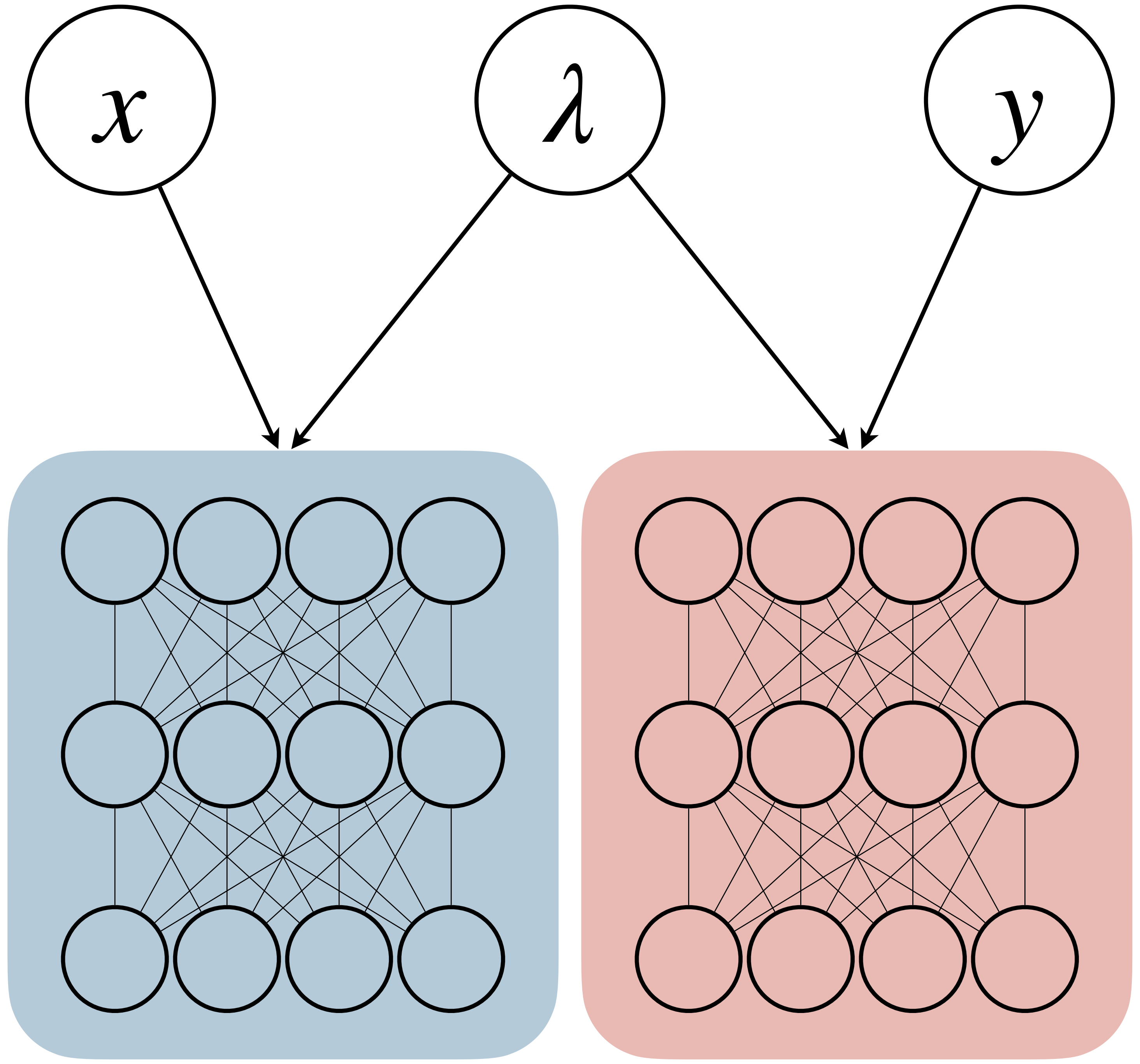

Model an LHV strategy with a Neural Network

- The parties will be represented by a pair of NN.
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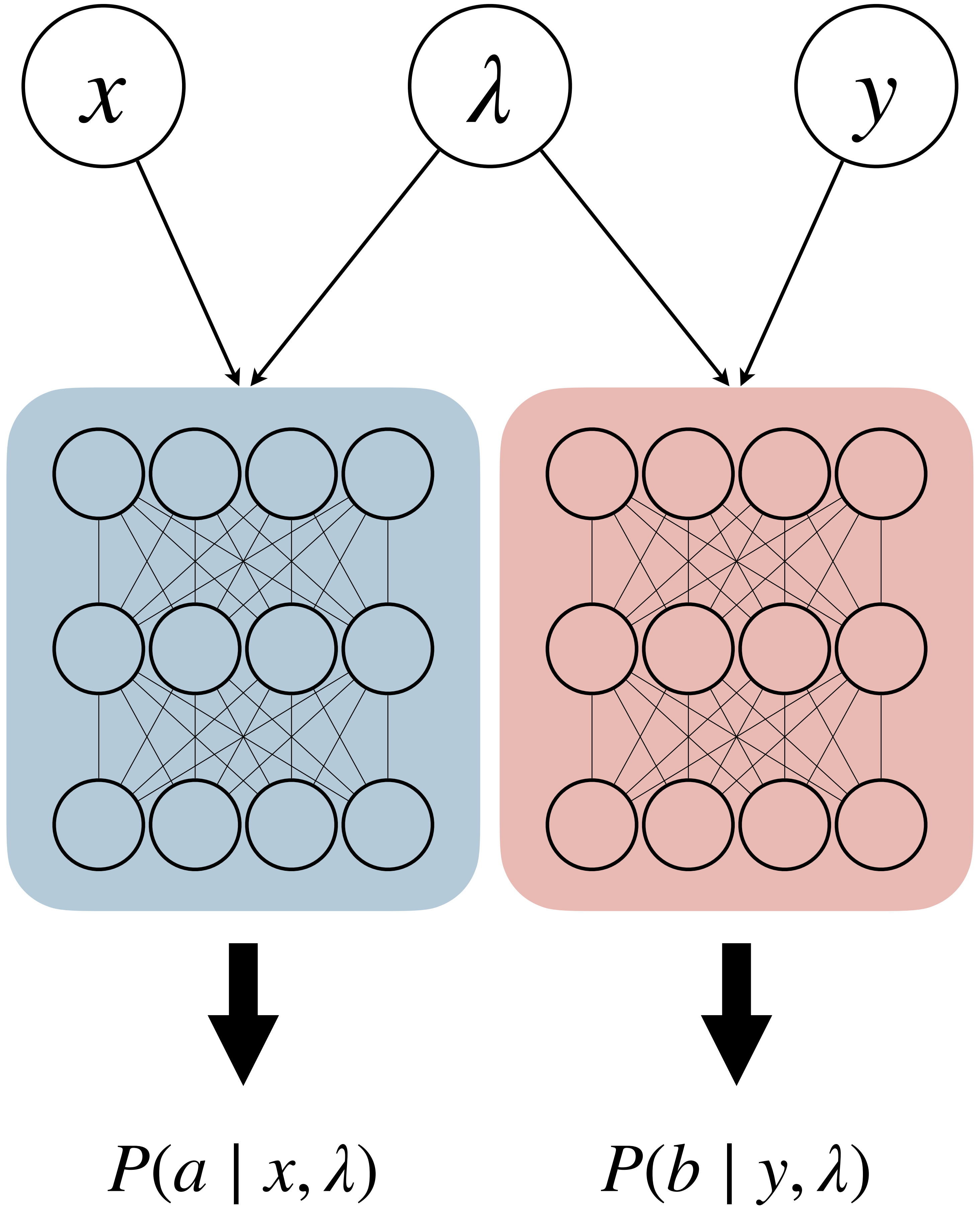
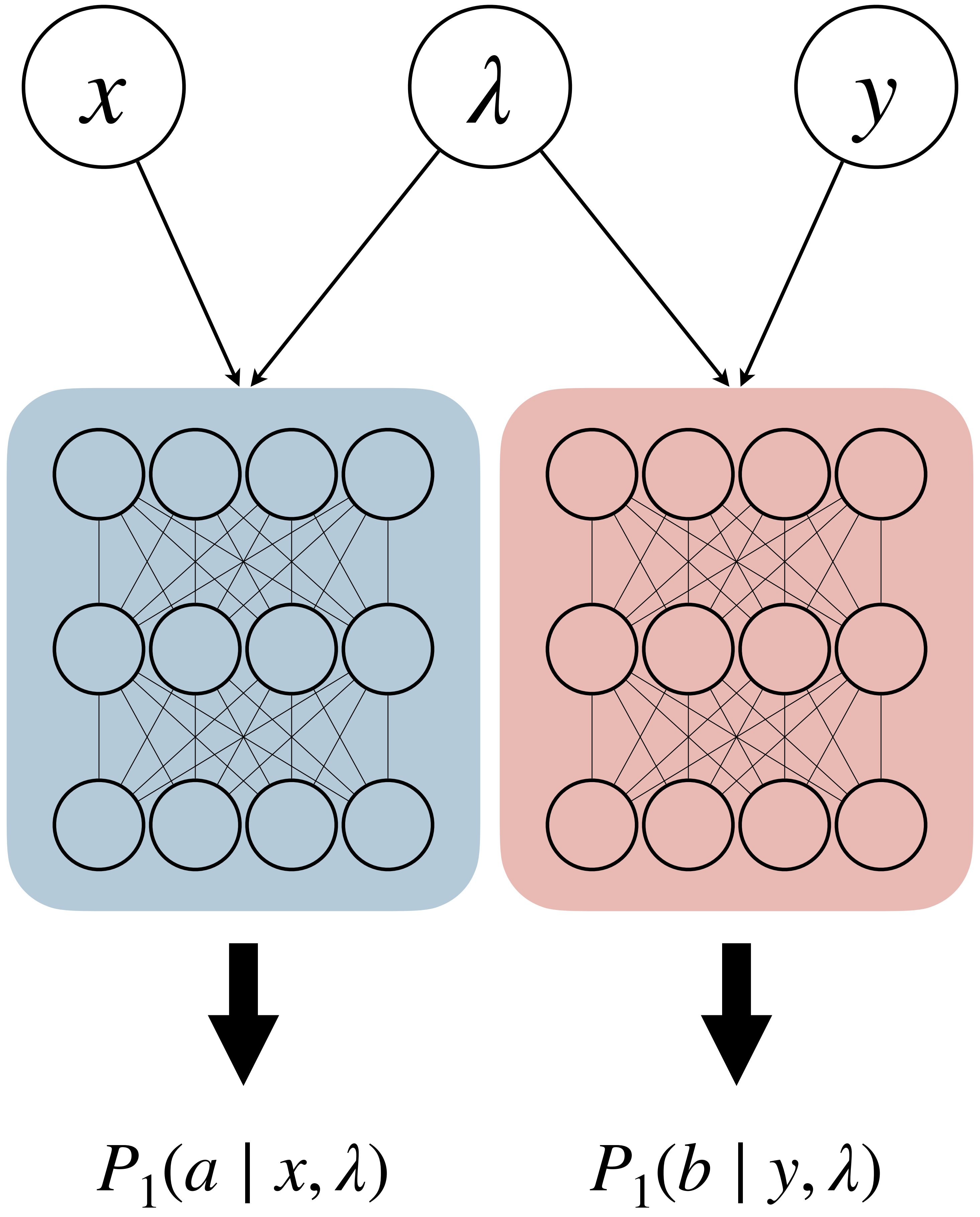
- $P(a, b \mid x, y)$ is the output after averaging over the LHV.



$$P(a, b \mid x, y) = \frac{1}{N} \sum_{\lambda}^{N} P(a \mid x, \lambda) P(b \mid y, \lambda)$$

Adding communication



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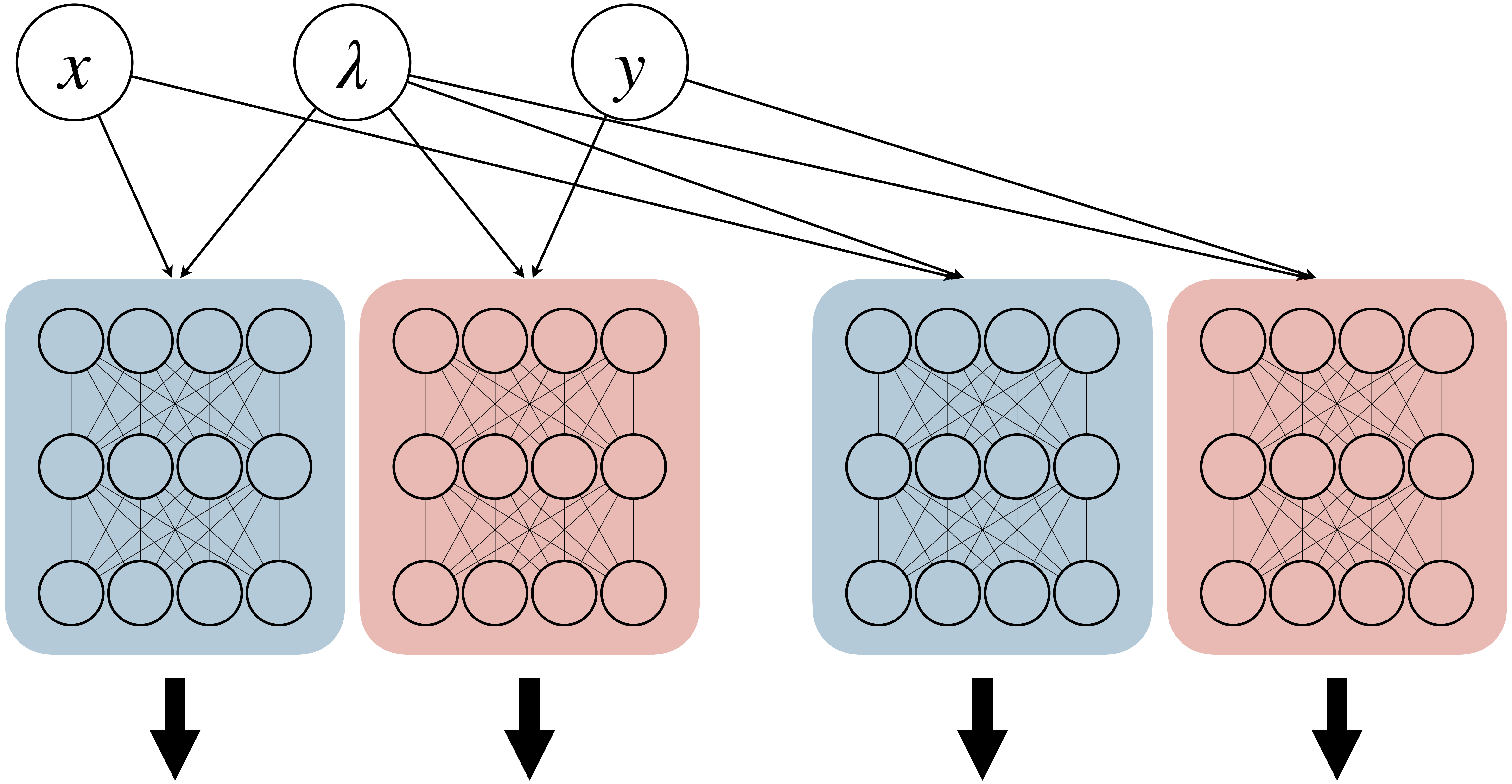
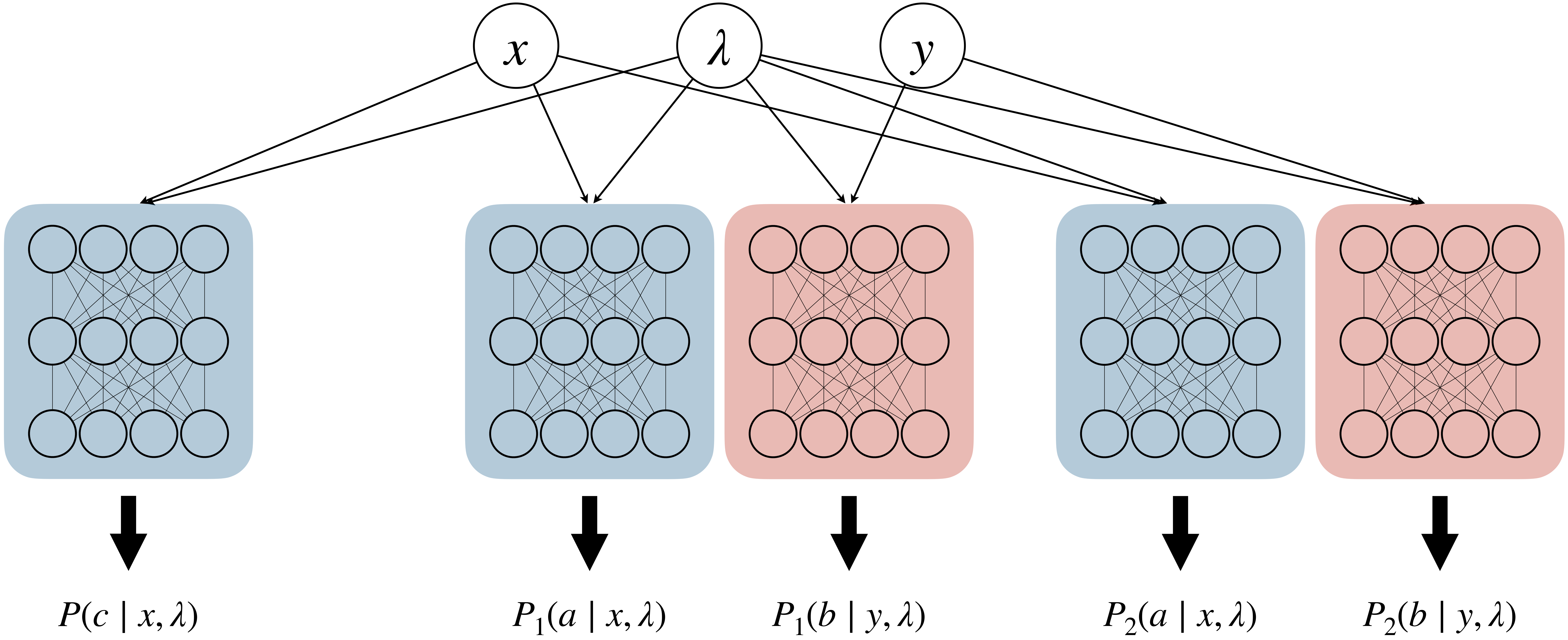
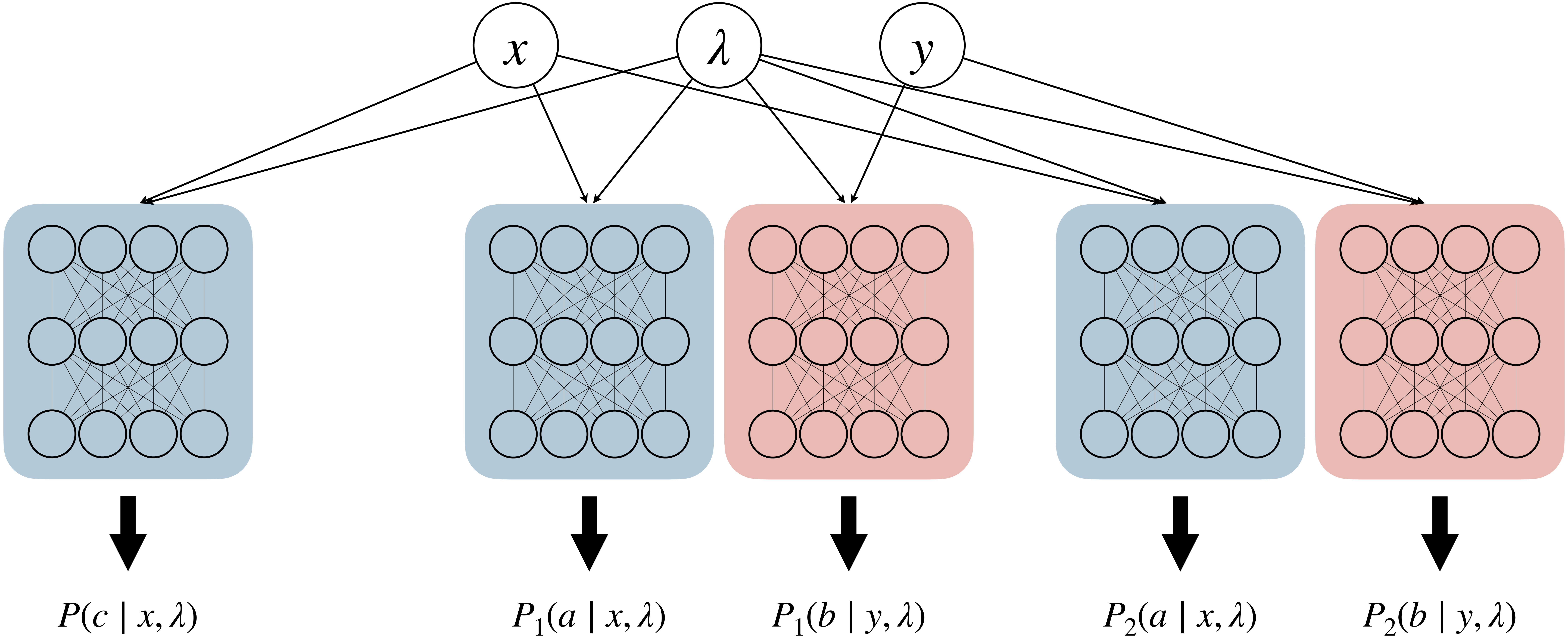


$P_1(a \mid x, \lambda)$ $P_1(b \mid y, \lambda)$ $P_2(a \mid x, \lambda)$ $P_2(b \mid y, \lambda)$

Adding communication





$$P(a,b \mid x,y) = \frac{1}{N}\sum_{\lambda}^{N} P(c=1 \mid x,\lambda)\, P_1(a \mid x,\lambda) P_1(b \mid y,\lambda) + (1 - P(c=1 \mid x,\lambda))\, P_2(a \mid x,\lambda) P_2(b \mid y,\lambda)$$

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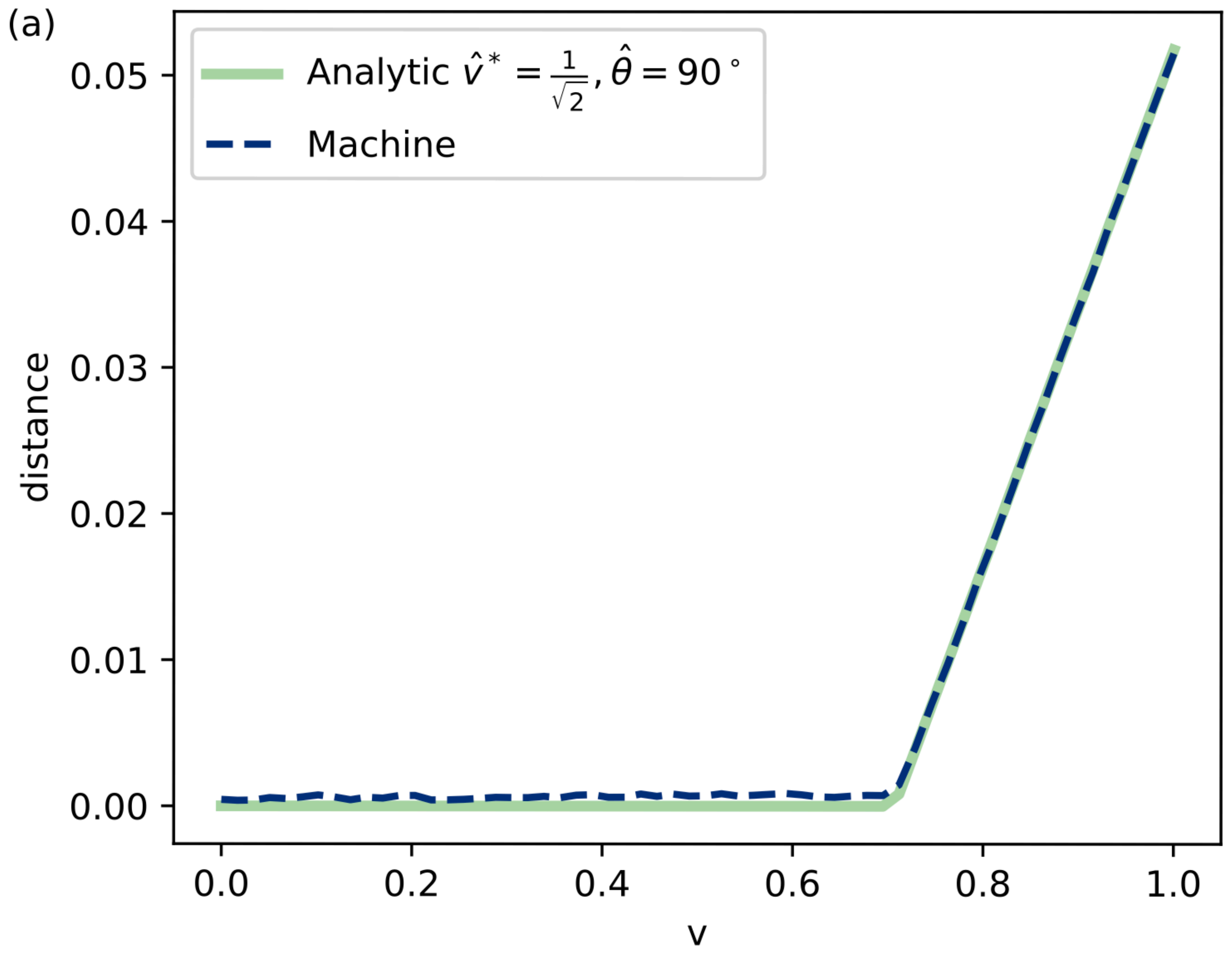

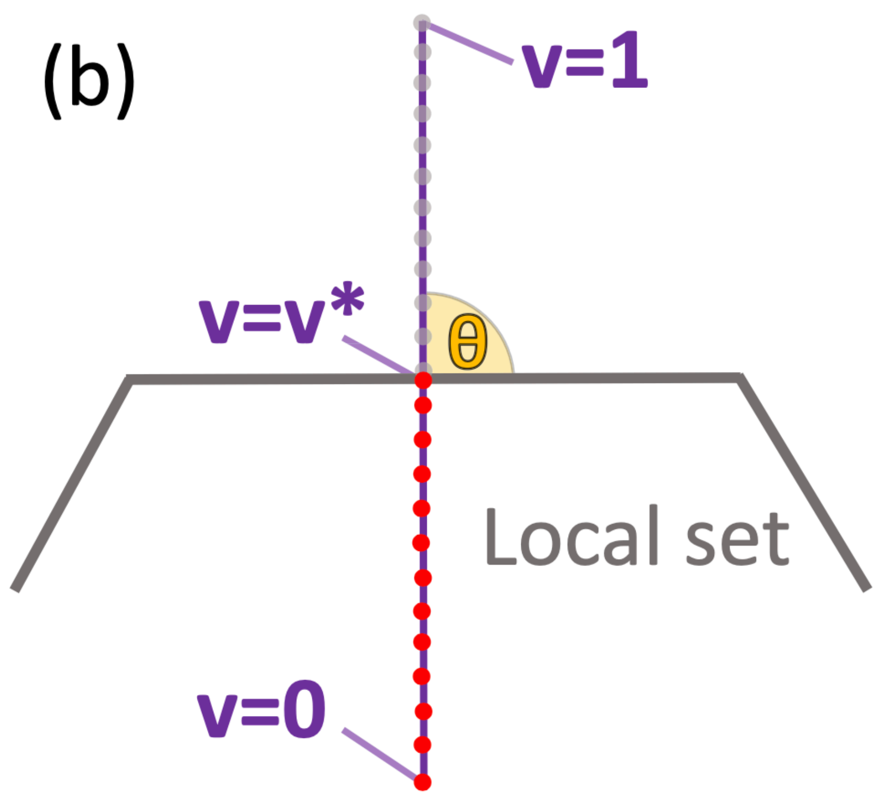

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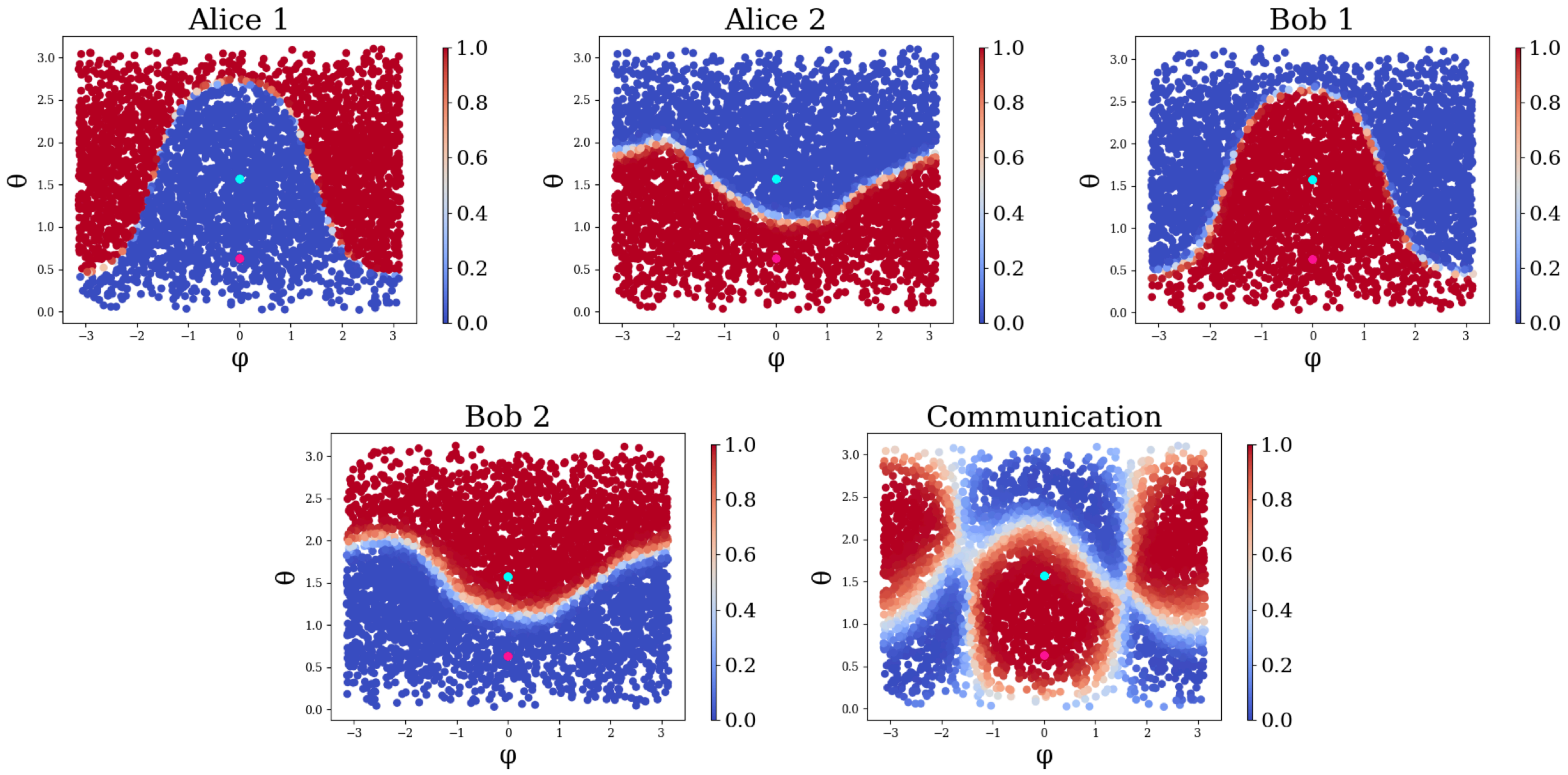
Why use ML?

- Neural network can be used to model LHV strategies.
- When it cannot achieve a given set of $P(a, b \mid x, y)$, it is a sign of a Bell inequality.
- On the other hand, if we train it over the set of all possible measurements, we can get a simulation protocol.

The results

Maximally entangled states



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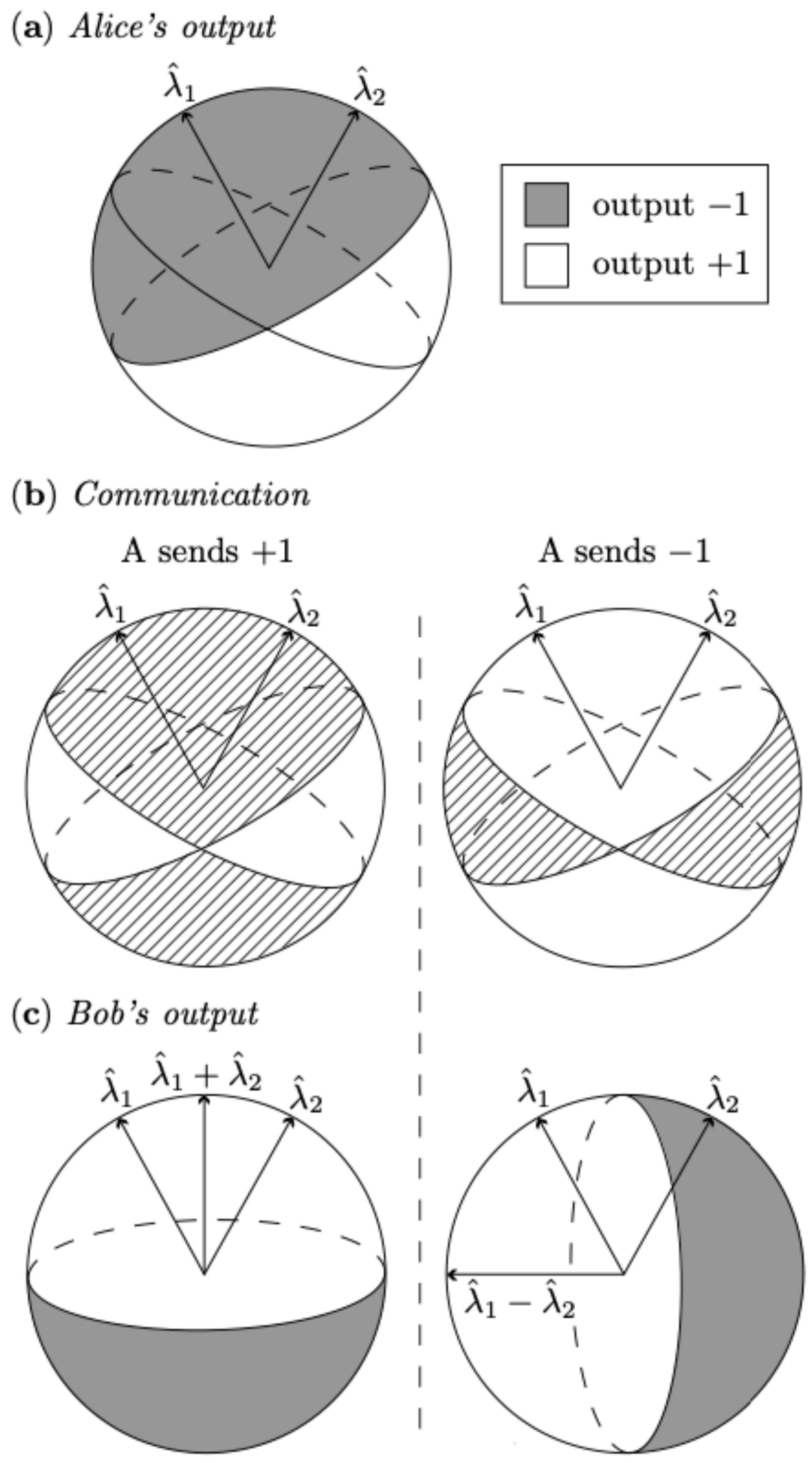

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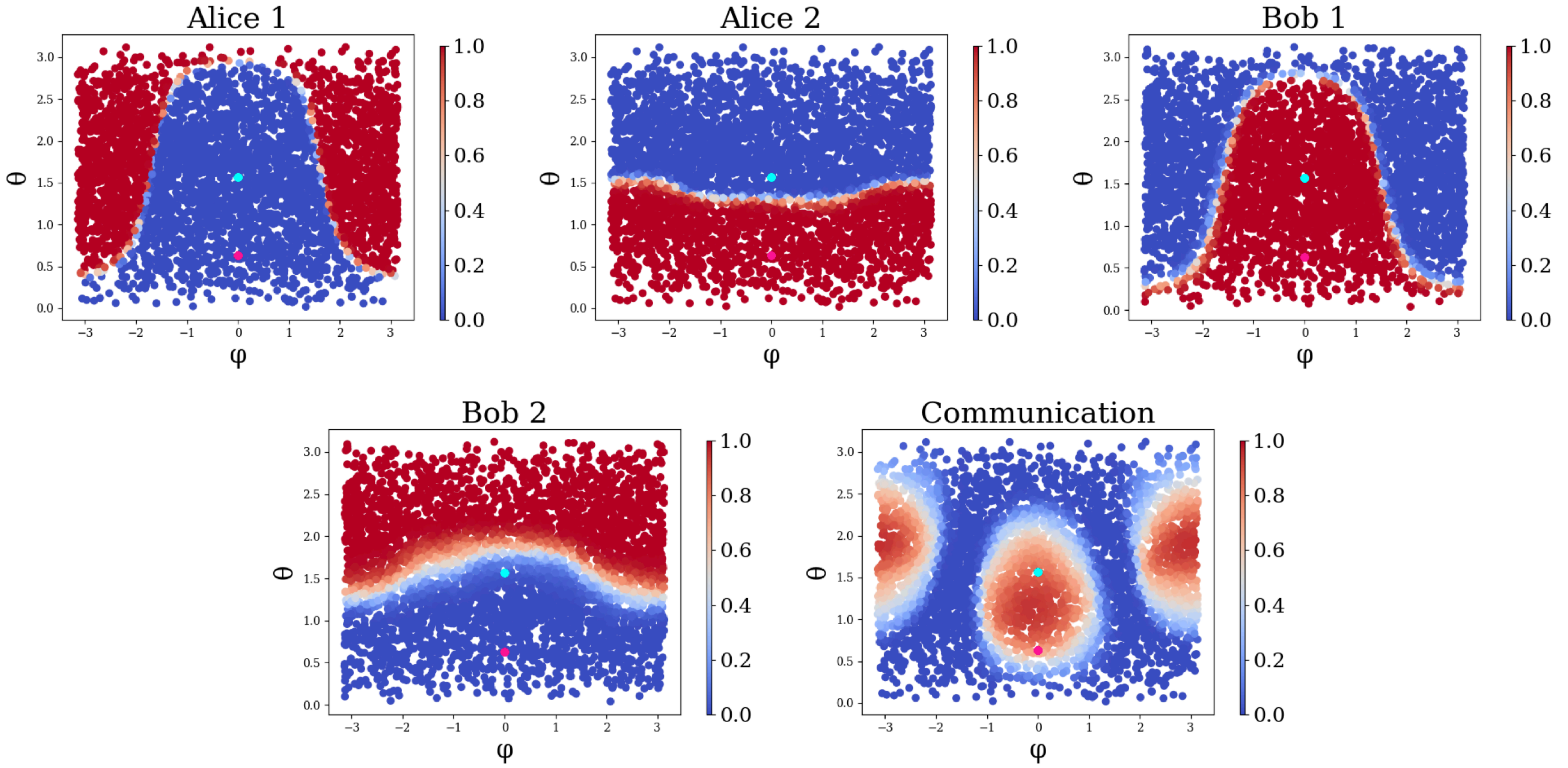
The results

Non-maximally entangled states

$$|\psi\rangle = \cos\left(\frac{5\pi}{32}\right)|00\rangle + \sin\left(\frac{5\pi}{32}\right)|11\rangle$$



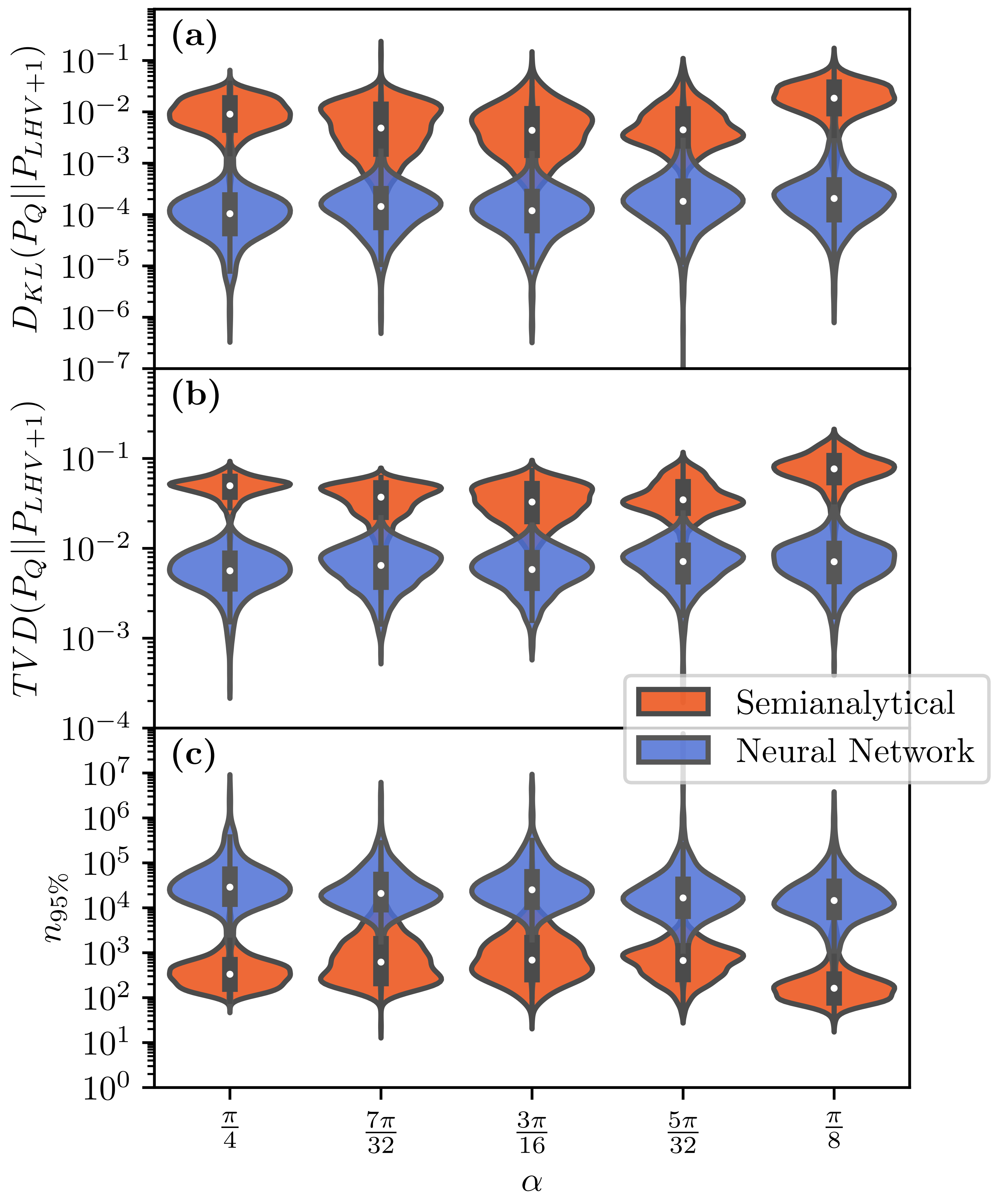
The results

Heuristic performance

Kullback-Leibler Divergence

Total Variational Distance

Sample size needed to differentiate



$$|\psi\rangle = \cos\alpha\,|00\rangle + \sin\alpha\,|11\rangle$$

Beating entanglement with 1-bit

Physical Review A 109 (6), 062408

Finding states that are unsimulatable

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So what states are unsimulatable with 1-bit?

The polytopes

Brunner, N., Cavalcanti, D., Pironio, S., Scarani, V. and Wehner, S., 2014. Bell nonlocality. *Reviews of modern physics*, *86*(2), pp.419-478. (figure)

The polytopes

- The set of all local behaviours forms a polytope L

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- The set of all local behaviours forms a polytope L
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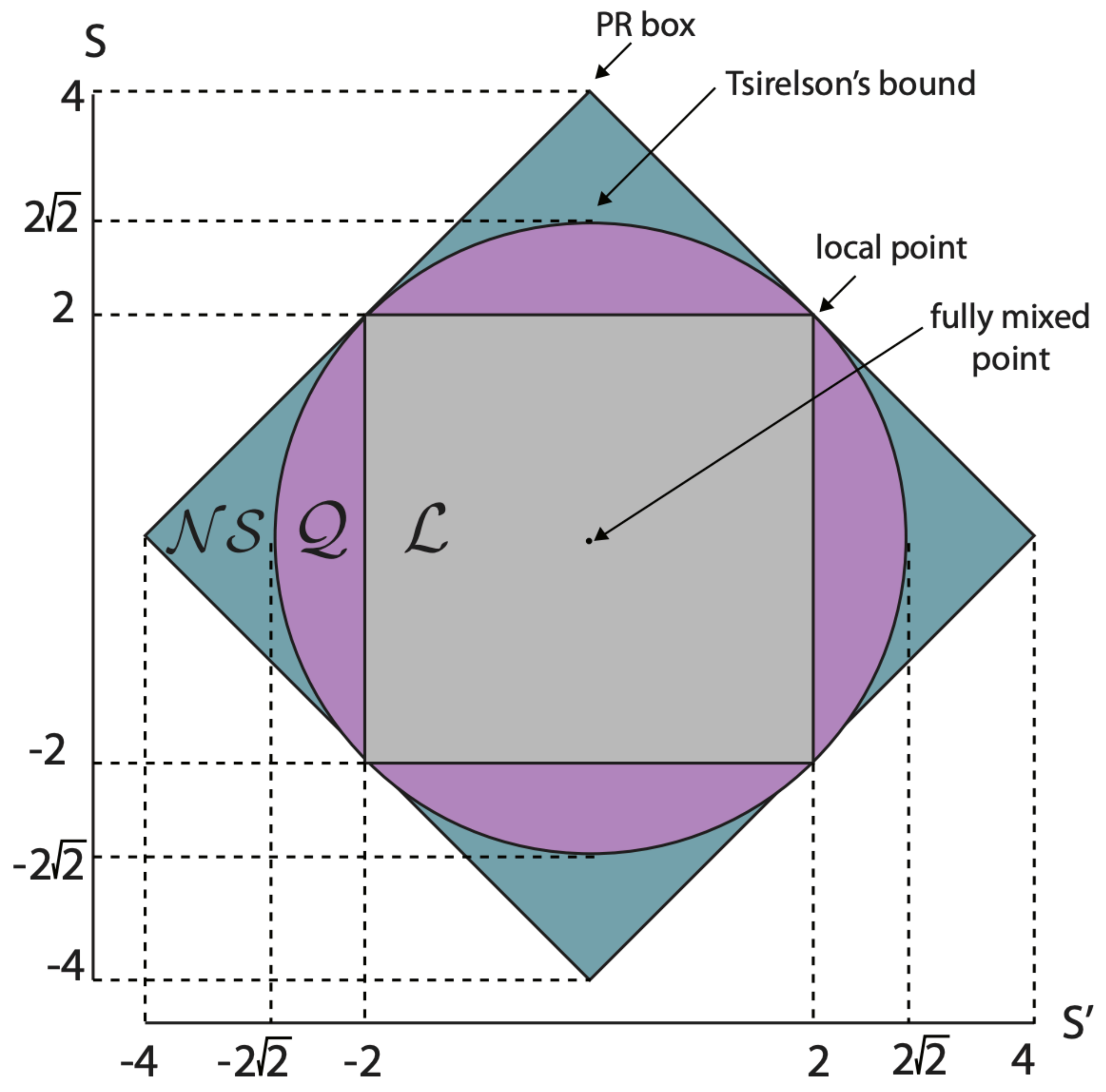

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Known results

L : local set $\quad$ C: 1 bit set $\quad$ Q: quantum set

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- Marton et al. showed that $Q \not\subset C$ in $(5, 3, 16, 16)$ with a Bell inequality for C.

Can we beat 1 bit in a smaller scenario?

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Vértesi, T. and Bene, E., 2009. Lower bound on the communication cost of simulating bipartite quantum correlations. *Physical Review A—Atomic, Molecular, and Optical Physics*, *80*(6), p.062316.
Márton, I., Bene, E., Diviánszky, P. and Vértesi, T., 2024. Beating one bit of communication with and without quantum pseudo-telepathy. *npj Quantum Information*, *10*(1), p.79.

Computing C

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- Facet enumeration of C is quickly infeasible.
- Parallel repetition results in large sizes of games.
- We devised a method to calculate the 1-bit bound S_C of a given Bell inequality quickly.

The notation

Zhen, Y.Z., Goh, K.T., Zheng, Y.L., Cao, W.F., Wu, X., Chen, K. and Scarani, V., 2016. Nonlocal games and optimal steering at the boundary of the quantum set. *Physical Review A*, *94*(2), p.022116. (figure)

The notation

- We can write a Bell inequality as

$$\sum_{a,b,x,y} V_{a,b,x,y} P(a, b \mid x, y) \leq S_L$$

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The notation

- We can write a Bell inequality as

$$\sum_{a,b,x,y} V_{a,b,x,y} P(a, b \mid x, y) \leq S_L$$

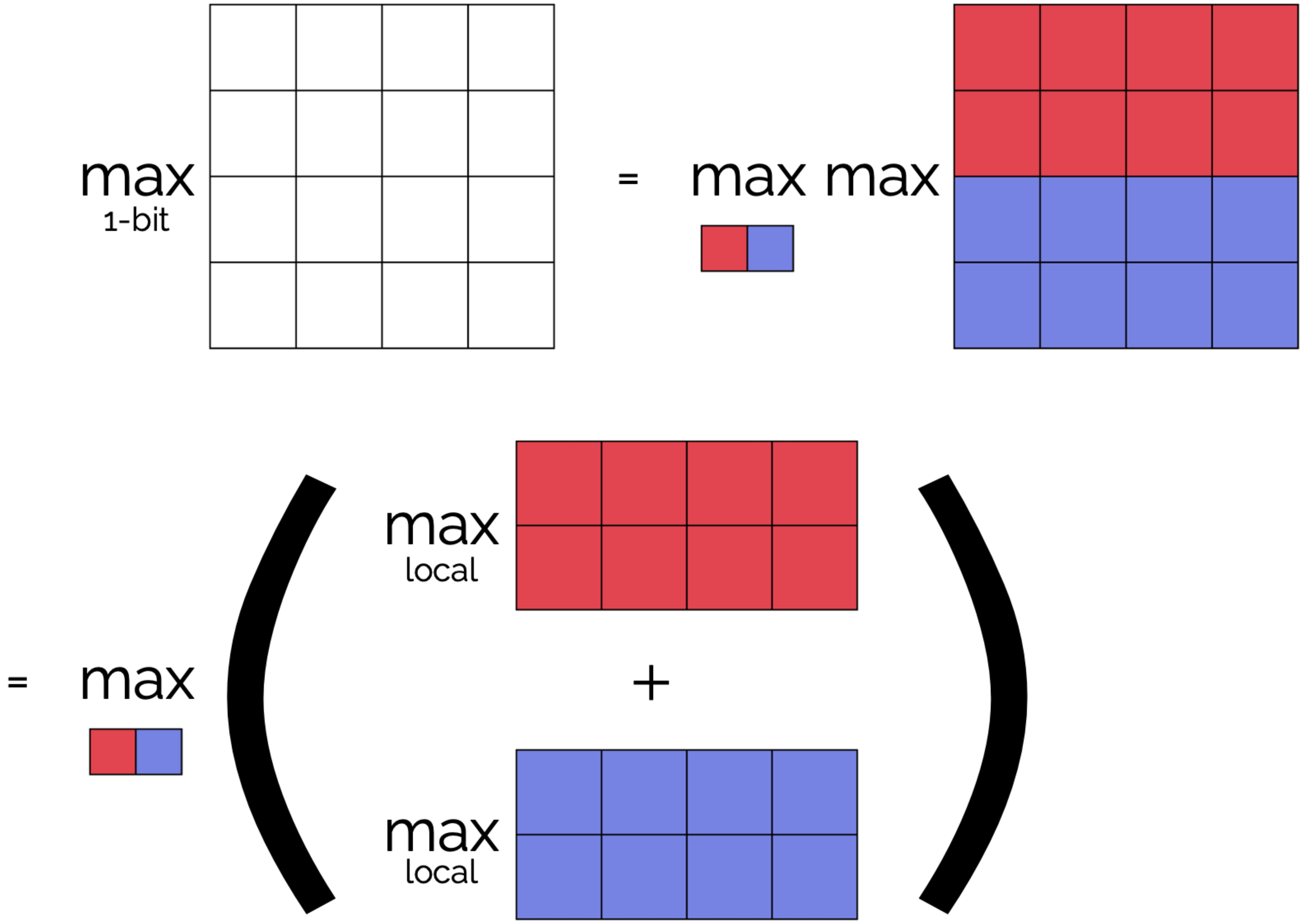
- We can write this out on a table

$$I = \begin{array}{|cc|cc|}\hline V_{0000} & V_{0100} & V_{0001} & V_{0101} \\ V_{1000} & V_{1100} & V_{1001} & V_{1101} \\ \hline V_{0010} & V_{0110} & V_{0011} & V_{0111} \\ V_{1010} & V_{1110} & V_{1011} & V_{1111} \\ \hline \end{array}$$

$$I_{CHSH} = \begin{array}{|cc|cc|}\hline 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ \hline \end{array}$$

Zhen, Y.Z., Goh, K.T., Zheng, Y.L., Cao, W.F., Wu, X., Chen, K. and Scarani, V., 2016. Nonlocal games and optimal steering at the boundary of the quantum set. *Physical Review A*, *94*(2), p.022116. (figure)

The method



Bell inequality for 1-bit of communication

	$y=1$					$y=2$				
$x=1$	1	·	·	·	·	1	·	·	·	·
	·	1	·	·	·	·	1	·	·	·
	·	·	1	·	·	·	·	1	·	·
	·	·	·	1	·	·	·	·	1	·
	·	·	·	·	1	·	·	·	·	1
$x=2$	1	·	·	·	·	·	1	·	·	·
	·	1	·	·	·	·	·	1	·	·
	·	·	1	·	·	·	·	·	1	·
	·	·	·	1	·	·	·	·	·	1
	·	·	·	·	1	1	·	·	·	·
$x=3$	1	·	·	·	·	·	·	1	·	·
	·	1	·	·	·	·	·	·	1	·
	·	·	1	·	·	·	·	·	·	1
	·	·	·	1	·	1	·	·	·	·
	·	·	·	·	1	·	1	·	·	·
$x=4$	1	·	·	·	·	·	·	·	1	·
	·	1	·	·	·	·	·	·	·	1
	·	·	1	·	·	1	·	·	·	·
	·	·	·	1	·	·	1	·	·	·
	·	·	·	·	1	·	·	1	·	·
$x=5$	1	·	·	·	·	·	·	·	·	1
	·	1	·	·	·	1	·	·	·	·
	·	·	1	·	·	·	1	·	·	·
	·	·	·	1	·	·	·	1	·	·
	·	·	·	·	1	·	·	·	1	·

Bell inequality for 1-bit of communication

- Truncated XOR game in 5d.

	$y=1$					$y=2$				
$x=1$	1	·	·	·	·	1	·	·	·	·
	·	1	·	·	·	·	1	·	·	·
	·	·	1	·	·	·	·	1	·	·
	·	·	·	1	·	·	·	·	1	·
	·	·	·	·	1	·	·	·	·	1
$x=2$	1	·	·	·	·	·	1	·	·	·
	·	1	·	·	·	·	·	1	·	·
	·	·	1	·	·	·	·	·	1	·
	·	·	·	1	·	·	·	·	·	1
	·	·	·	·	1	1	·	·	·	·
$x=3$	1	·	·	·	·	·	·	1	·	·
	·	1	·	·	·	·	·	·	1	·
	·	·	1	·	·	·	·	·	·	1
	·	·	·	1	·	1	·	·	·	·
	·	·	·	·	1	·	1	·	·	·
$x=4$	1	·	·	·	·	·	·	·	1	·
	·	1	·	·	·	·	·	·	·	1
	·	·	1	·	·	1	·	·	·	·
	·	·	·	1	·	·	1	·	·	·
	·	·	·	·	1	·	·	1	·	·
$x=5$	1	·	·	·	·	·	·	·	·	1
	·	1	·	·	·	1	·	·	·	·
	·	·	1	·	·	·	1	·	·	·
	·	·	·	1	·	·	·	1	·	·
	·	·	·	·	1	·	·	·	1	·

Bell inequality for 1-bit of communication

- Truncated XOR game in 5d.
- Local score $S_L = 6$

	$y=1$					$y=2$				
	1	·	·	·	·	1	·	·	·	·
	·	1	·	·	·	·	1	·	·	·
$x=1$	·	·	1	·	·	·	·	1	·	·
	·	·	·	1	·	·	·	·	1	·
	·	·	·	·	1	·	·	·	·	1
	1	·	·	·	·	·	1	·	·	·
	·	1	·	·	·	·	·	1	·	·
$x=2$	·	·	1	·	·	·	·	·	1	·
	·	·	·	1	·	·	·	·	·	1
	·	·	·	·	1	1	·	·	·	·
	1	·	·	·	·	·	·	1	·	·
	·	1	·	·	·	·	·	·	1	·
$x=3$	·	·	1	·	·	·	·	·	·	1
	·	·	·	1	·	1	·	·	·	·
	·	·	·	·	1	·	1	·	·	·
	1	·	·	·	·	·	·	·	1	·
	·	1	·	·	·	·	·	·	·	1
$x=4$	·	·	1	·	·	1	·	·	·	·
	·	·	·	1	·	·	1	·	·	·
	·	·	·	·	1	·	·	1	·	·
	1	·	·	·	·	·	·	·	·	1
	·	1	·	·	·	1	·	·	·	·
$x=5$	·	·	1	·	·	·	1	·	·	·
	·	·	·	1	·	·	·	1	·	·
	·	·	·	·	1	·	·	·	1	·

Bell inequality for 1-bit of communication

- Truncated XOR game in 5d.
- Local score $S_L = 6$
- Quantum score $7.1777 < S_Q < 7.1788$

	$y = 1$	$y = 2$
$x = 1$	1 · · · ·	1 · · · ·
	· 1 · · ·	· 1 · · ·
	· · 1 · ·	· · 1 · ·
	· · · 1 ·	· · · 1 ·
	· · · · 1	· · · · 1
$x = 2$	1 · · · ·	· 1 · · ·
	· 1 · · ·	· · 1 · ·
	· · 1 · ·	· · · 1 ·
	· · · 1 ·	· · · · 1
	· · · · 1	1 · · · ·
$x = 3$	1 · · · ·	· · 1 · ·
	· 1 · · ·	· · · 1 ·
	· · 1 · ·	· · · · 1
	· · · 1 ·	1 · · · ·
	· · · · 1	· 1 · · ·
$x = 4$	1 · · · ·	· · · 1 ·
	· 1 · · ·	· · · · 1
	· · 1 · ·	1 · · · ·
	· · · 1 ·	· 1 · · ·
	· · · · 1	· · 1 · ·
$x = 5$	1 · · · ·	· · · · 1
	· 1 · · ·	1 · · · ·
	· · 1 · ·	· 1 · · ·
	· · · 1 ·	· · 1 · ·
	· · · · 1	· · · 1 ·

Bell inequality for 1-bit of communication

- Truncated XOR game in 5d.
- Local score $S_L = 6$
- Quantum score $7.1777 < S_Q < 7.1788$
- 1 bit score $S_C = 7$

	$y=1$	$y=2$
$x=1$	$\begin{matrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{matrix}$	$\begin{matrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{matrix}$
$x=2$	$\begin{matrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{matrix}$	$\begin{matrix} \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot \end{matrix}$
$x=3$	$\begin{matrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{matrix}$	$\begin{matrix} \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \end{matrix}$
$x=4$	$\begin{matrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{matrix}$	$\begin{matrix} \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \end{matrix}$
$x=5$	$\begin{matrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{matrix}$	$\begin{matrix} \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \end{matrix}$

Conclusion

- Two-qubits states can be closely approximated with one bit of communication.
- Two-qu-5-its states cannot be simulated with one bit of communication. This is shown with a (5,2,5,5) inequality.

Fin

The outputs of Alice are of the form of

$$P(A_1 = +1 \mid \hat{a}) = \frac{1}{2}(1 - \mathrm{sgn}(\hat{a} \cdot \vec{\lambda}_{a1} + b_{a1})),$$

where $\hat{\lambda}_{a1} = u_{a1}\vec{\lambda}_1 + \vec{\lambda}_2 + v_{a1}\hat{z}$ decides the hemisphere direction and $b_{a1} = w_{a1} + x_{a1}\vec{\lambda}_1 \cdot \hat{z} + y_{a1}\vec{\lambda}_2 \cdot \hat{z}$ decides the size of the hemisphere. Similarly,

$$P(A_2 = +1 \mid \hat{a}) = \frac{1}{2}(1 + \mathrm{sgn}(\hat{a} \cdot \vec{\lambda}_{a2} + b_{a2})),$$

$$P(B_1 = +1 \mid \hat{b}) = \frac{1}{2}(1 + \mathrm{sgn}(\hat{b} \cdot \vec{\lambda}_{b1} + b_{b1})),$$

$$P(B_2 = +1 \mid \hat{b}) = \frac{1}{2}(1 - \mathrm{sgn}(\hat{b} \cdot \vec{\lambda}_{b2} + b_{b2})).$$

Using numerical algorithms, we can approximately obtain the relevant coefficients for the different states.

The (simplified) bit of communication is given by

$$P(c = +1 \mid \hat{a}) = \frac{1}{2}(1 - \mathrm{clip}(f_c, -1, 1)),$$

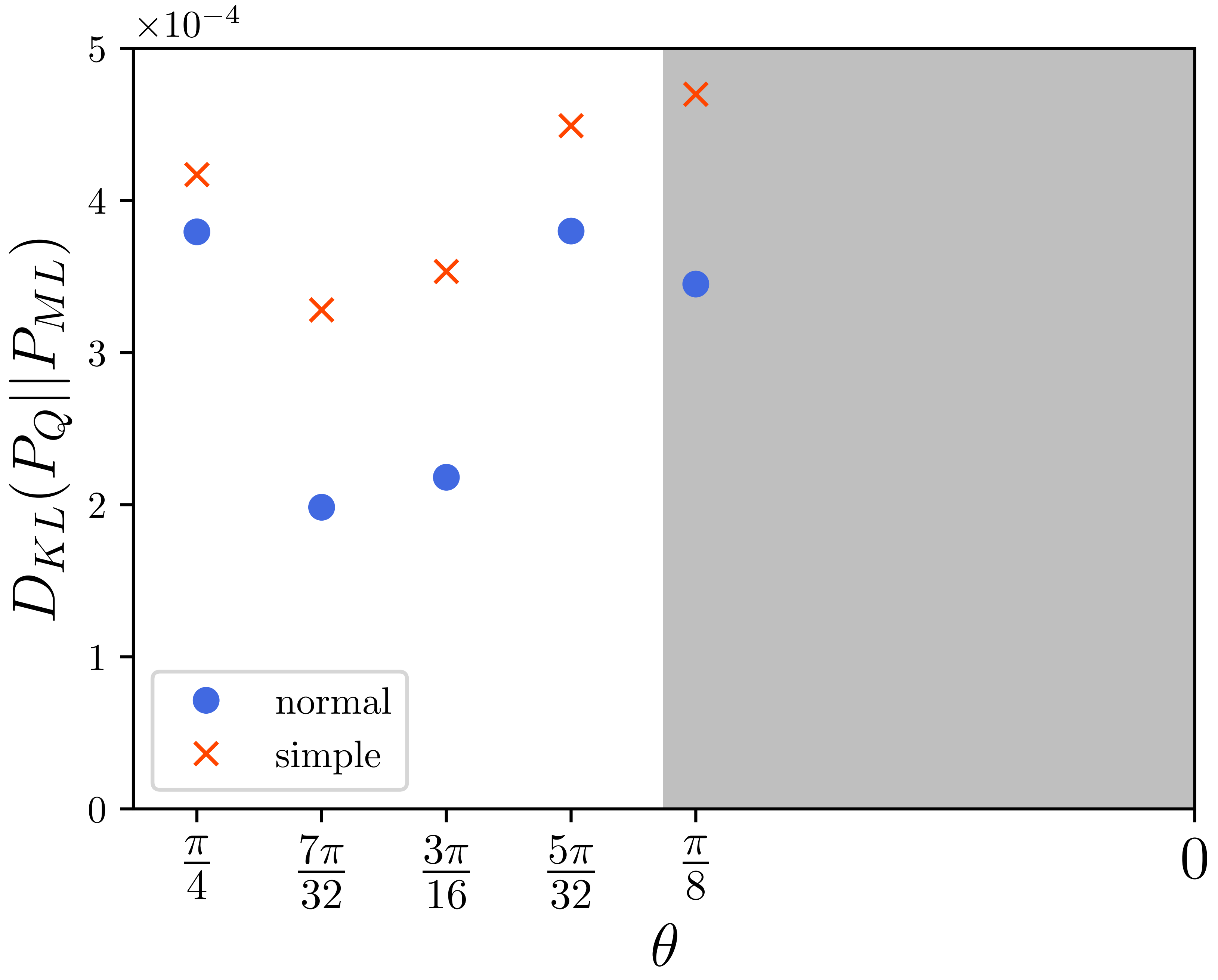
where

$$\begin{aligned} f_c = {} & \Theta(\hat{a} \cdot \vec{\lambda}_1 + b_c)\Theta(\hat{a} \cdot \vec{\lambda}_2 + b_c) \\ & + \Theta(-\hat{a} \cdot \vec{\lambda}_1 + b_c)\Theta(-\hat{a} \cdot \vec{\lambda}_2 + b_c) \\ & - \Theta(-\hat{a} \cdot \vec{\lambda}_1 - b_c)\Theta(\hat{a} \cdot \vec{\lambda}_2 - b_c) \\ & - \Theta(\hat{a} \cdot \vec{\lambda}_1 - b_c)\Theta(-\hat{a} \cdot \vec{\lambda}_2 - b_c), \end{aligned}$$

with $b_c = u_c + v_c(\vec{\lambda}_2 \cdot \hat{z})(1 - \vec{\lambda}_1 \cdot \hat{z})$ and the clip function is defined as

$$\mathrm{clip}(x, a, b) = \begin{cases} a & \text{if } x < a \\ b & \text{if } x > b \\ x & \text{otherwise} \end{cases}.$$

Again, the relevant coefficients are obtained using numerical methods.



×10^−4
5
4
3
2
1
0
$D_{KL}(P_Q||P_{ML})$
normal
simple
π/4
7π/32
3π/16
5π/32
π/8
0
θ