Correlation and simulatibility transitions in monitored dynamics of matrix product states

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Outline of the presentation

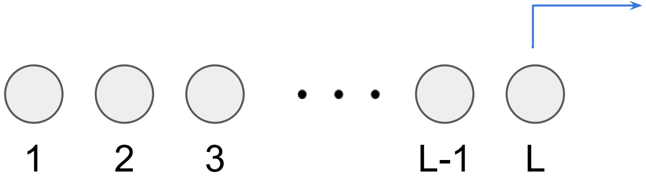
1. Entanglement and classical simulatibility
2. Matrix product states (MPSs)
3. Hybrid dynamics measurement-induced phase transitions (MIPTs)
4. Capturing the MIPT with MPS
    a. Time-dependent variational principle (TDVP)
    b. The time-dependent variational principle
    c. MIPT as a transition in classical simulatibility
5. Correlations in ensembles of MPS
    a. Random matrix product state (RMPS)
    b. Mutual information in RMPs
    c. Mutual information in monitored systems

Classical simulatability and MPS

When is a quantum system **not classically** simulatable?

Let us take the **prototypical** quantum system

1D **qubit** lattice (spin chain)



Local Hilbert space

$$\{|\uparrow\rangle, |\downarrow\rangle\}$$

The total Hilbert space is **exponentially** large → $D = 2^L$

But that is not enough…

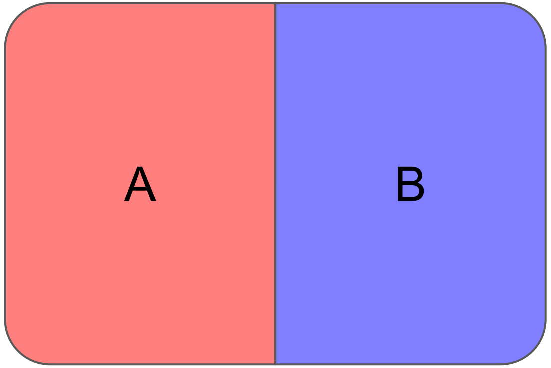
- Free fermionic models (Gaussian, non-interacting)
- Clifford circuits
- **Matrix product states** (when efficiently compressed)
- …

} Simulate with polynomial complexity in L

A necessary (but not sufficient) condition → **Quantum entanglement**

How to **quantify** quantum entanglement?

An entangled system cannot be described by the state of its components in isolation



$$\rho_A = \mathrm{Tr}_B(\rho)$$

Why are entanglement laws connected to classical simulatability?

Rényi entropy

$$S_A^n = \frac{1}{1-n} \log\left[\mathrm{Tr}\left(\rho_A^n\right)\right]$$

Von Neumann entanglement entropy

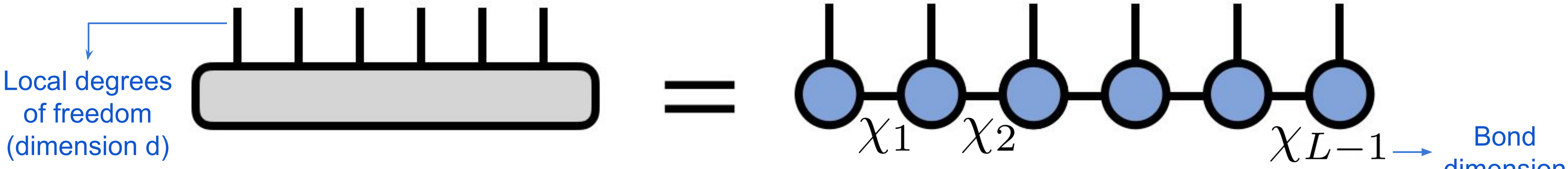
$$S_A^1 = S_A = -\mathrm{Tr}\left(\rho_A \log \rho_A\right)$$

Is entanglement **extensive**?

$$S_A \propto \quad V_A \quad \log V_A \quad V_{\partial A}$$

V_A	$\log V_A$	$V_{\partial A}$
Volume law	**Log (critical)** law	**Area** law

Quick introduction to **matrix product states**

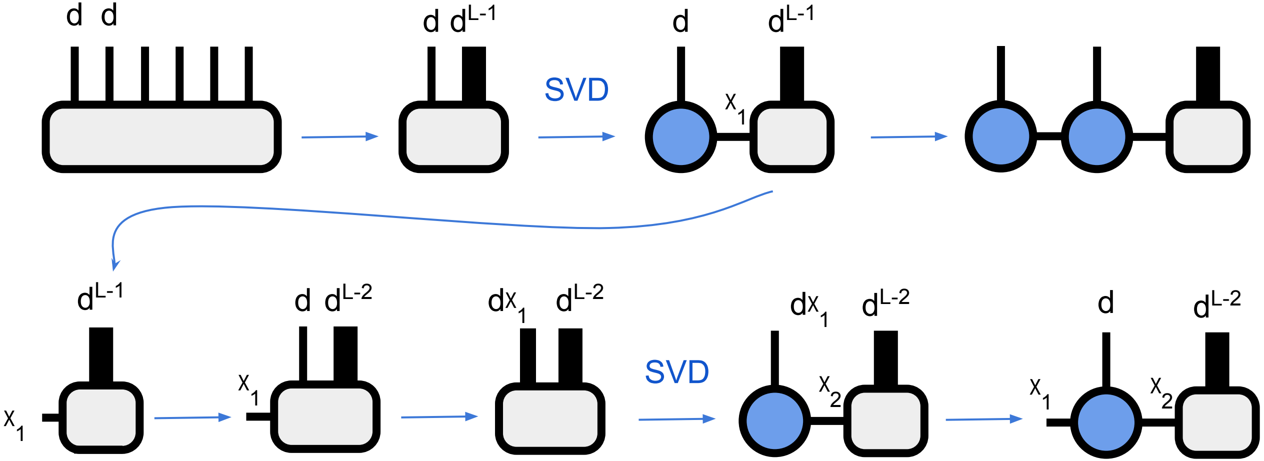


Local degrees of freedom (dimension d)

Bond dimension

$$|\Psi(M)\rangle = \sum_{\sigma_1,\dots,\sigma_L} M^{\sigma_1}_{1;\chi_1} \cdots M^{\sigma_L}_{L;\chi_{L-1}} |\sigma_1 \cdots \sigma_L\rangle$$

How to **construct** the MPS?



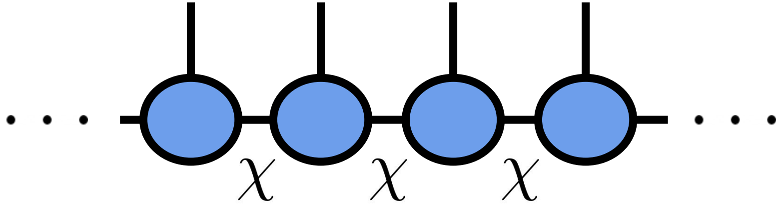
Exponential growth of the bond dimension

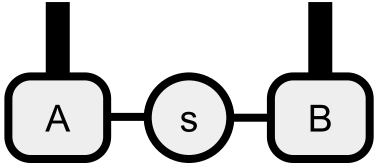
$$\chi_1 = d$$
$$\chi_2 = d^2$$
$$\vdots$$
$$\chi_n = d^n$$

Can we **compress** the MPS?

Fix maximal **bond dimension** $\chi_i \leqslant \chi$

(truncate smaller Schmidt values after SVD)





$$|\psi\rangle = \sum_i s_i \left|\psi_i^A\right\rangle \otimes \left|\psi_i^B\right\rangle \Rightarrow S_A = -\sum_i s_i^2 \log s_i^2$$

Entanglement **upper bound** at fixed bond dimension χ $\quad S_A \leq \log \chi$

There is always a **finite** χ such that

$$\| |\psi_{\mathrm{Area}}\rangle - |\psi_{\mathrm{MPS}}\rangle \|^2 < \varepsilon \quad \forall \quad \varepsilon > 0$$

MPS can be **efficiently** compressed
for **area law** states

Efficient numerical simulations using matrix-product states, Frank Pollmann (2016)

Hybrid circuits and MIPTs

Entanglement growth in **quantum circuits**

Why quantum circuits?

- Fundamental in quantum computing
- **Trotterization** of local Hamiltonian evolution

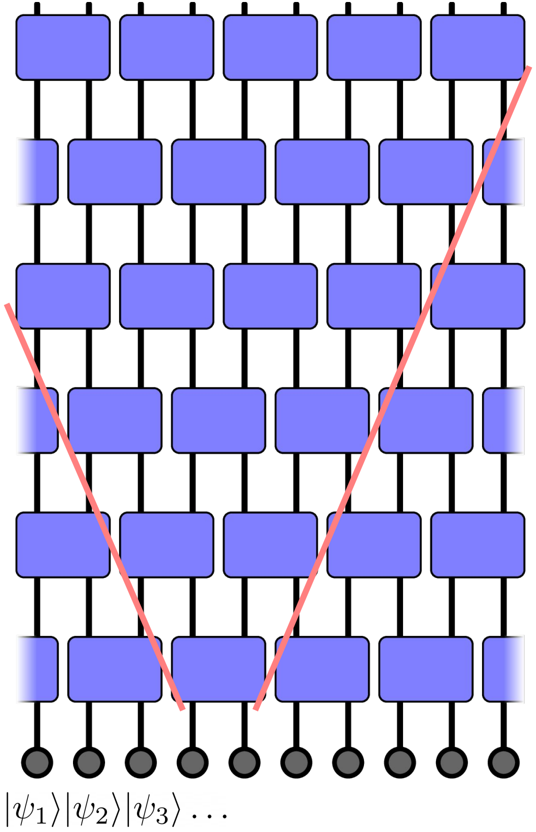
$$\hat{H} = \sum_i \hat{h}_{i,i+1}$$

First order Suzuki–Trotter expansion

$$\begin{aligned}\hat{U} &= e^{-i\delta t \hat{H}} \\ &= e^{-i\delta t \hat{H}_{\text{odd}}} e^{-i\delta t \hat{H}_{\text{even}}} + \mathcal{O}(\delta t^2)\end{aligned}$$

Ballistic growth of entanglement until saturation at **volume**-law

Unitary circuit



Evolution with random **monitoring**

With probability $p = \gamma\delta t$

└→ Measurement rate

Measure operator $\hat{O} = \sum_k o_k \hat{P}_k$ with the **Born rule**

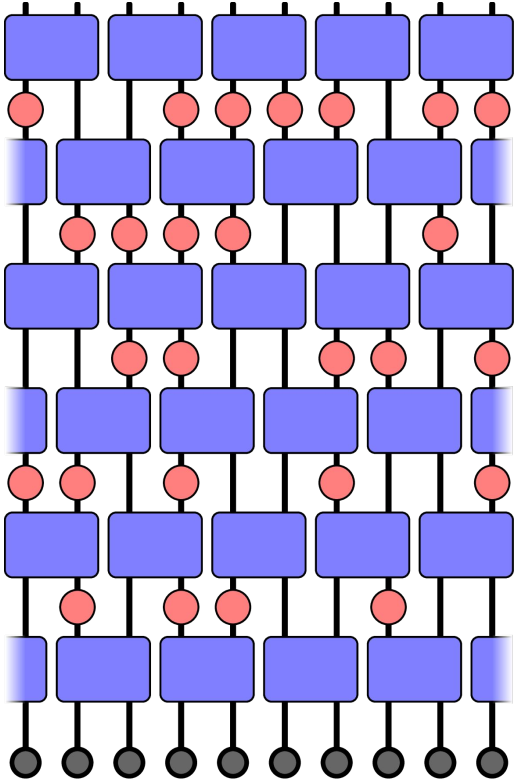
Projective (strong) measurements

$|\psi\rangle \rightarrow \frac{\hat{P}_k|\psi\rangle}{\sqrt{p_k}}$ with probability $p_k = \langle\psi|\hat{P}_k|\psi\rangle$

Average entanglement over **quantum trajectories**

Competition between entangling **unitary** gates and disentangling **measurements**

Hybrid circuit

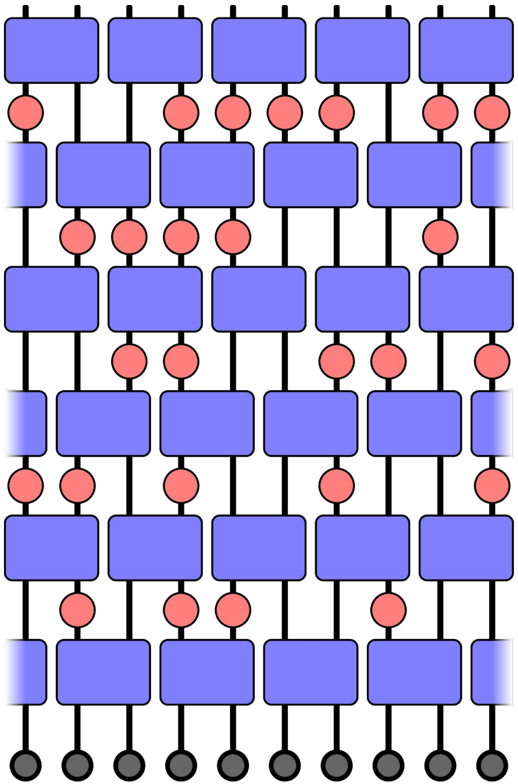


Measurement-induced **entanglement** phase transition

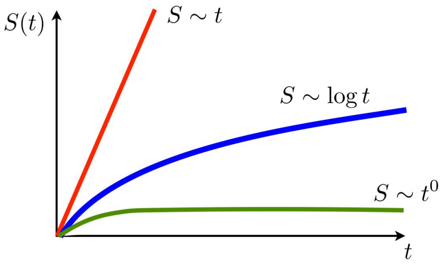
Trajectory-averaged **nonlinear** functions of the state → Measurement-induced phase transitions (MIPTs)

$$S(A) = -\mathrm{Tr}(\rho_A \log \rho_A)$$

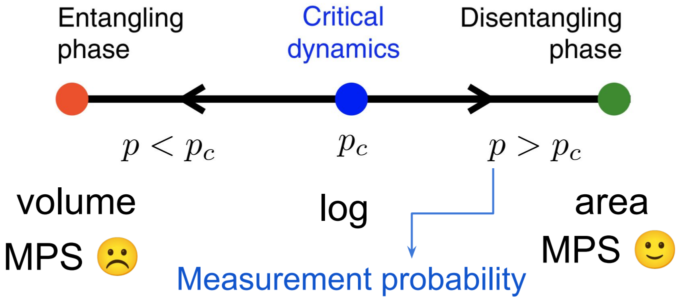
Hybrid circuit



Entanglement growth



Saturated entanglement laws



Entangling phase

Critical dynamics

Disentangling phase

$p < p_c$

p_c

$p > p_c$

volume

MPS ☹️

log

area

MPS 🙂

Measurement probability

Does a truncated MPS still retain information about the MIPT?

Measurement-Induced Phase Transitions in the Dynamics of Entanglement, Brian Skinner, Jonathan Ruhman, and Adam Nahum, Phys. Rev. X 9, 031009 (2019)

Measurement-driven entanglement transition in hybrid quantum circuits, Yaodong Li, Xiao Chen, and Matthew P. A. Fisher, Phys. Rev. B 100, 134306 (2019)

Probing the MIPT with MPS

The Time-Dependent Variational Principle (**TDVP**)

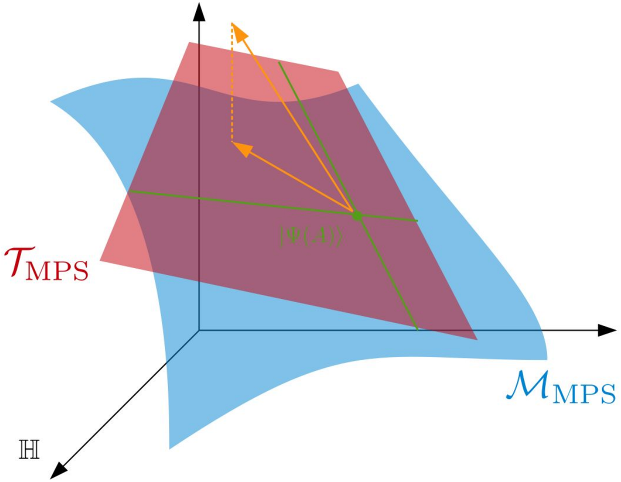
Time evolution with **projection** to MPS manifold

$$i\partial_t|\Psi(M)\rangle = P_{\mathcal{T}_M}\hat{H}|\Psi(M)\rangle$$

Tangent space of fixed χ

Effective nonlinear **symplectic** evolution
(unitary within the manifold)

Have the same **conservation** laws of the exact dynamics



Error rate $$E(\chi) = \|\hat{H}\,|\psi\rangle - P_{\mathcal{T}_{M_\chi}}\hat{H}\,|\psi\rangle\,\|^2$$

Volume law → $E(\chi) \sim \dfrac{1}{\log(\chi)}$

Area law → $E(\chi) \sim e^{-\chi}$

Unifying time evolution and optimization with matrix product states, Jutho Haegeman, Christian Lubich, Ivan Oseledets, Bart Vandereycken, Frank Verstraete, arXiv:1408.5056 (2015)

$\mathrm{U}(1)$ symmetric **interacting** model

Example: **XXX** spin chain
$$\hat{H}_{\mathrm{XXX}} = \sum_{i=1}^{L}\left(\hat{S}_i^x\hat{S}_{i+1}^x + \hat{S}_i^y\hat{S}_{i+1}^y + \hat{S}_i^z\hat{S}_{i+1}^z\right)$$

Measure local magnetization $\hat{S}_i^z$ with a measurement rate γ

Continuous measurements - **stochastic Schödinger equation**

$$d\,|\psi_t\rangle = -iH\,dt\,|\psi_t\rangle \; + \sum_{i=1}^{L}\left[\sqrt{\gamma}(\hat{S}_i^z - \langle\hat{S}_i^z\rangle_t)dW_t^i - \frac{\gamma}{2}(\hat{S}_i^z - \langle\hat{S}_i^z\rangle_t)^2 dt\right]|\psi_t\rangle$$

$$\left|\psi_{t+\delta t}\right\rangle \approx C e^{\sum_{j=1}^{L}\left[\delta W_t^j + 2\left\langle\hat{S}_j^z\right\rangle_t\gamma\delta t\right]\hat{S}_j^z} e^{-\mathbf{i}\hat{H}\delta t}\left|\psi_t\right\rangle \qquad \delta W_t^j \sim \mathcal{N}(0,\gamma\delta t)$$

Single site operators,
no truncation

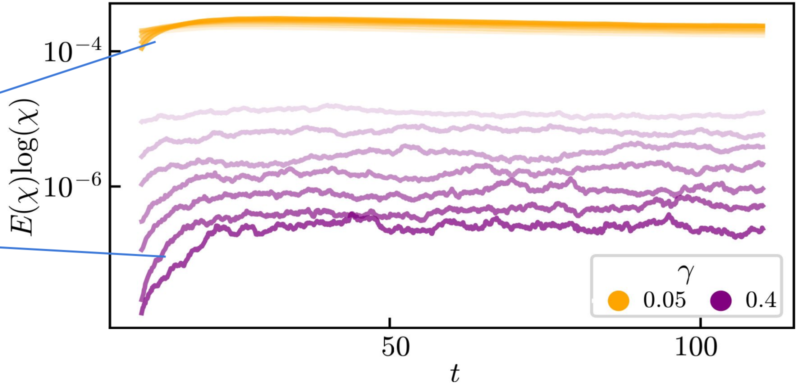
TDVP
algorithm

Probing the **entanglement** transition with the TDVP error

Large time behaviour of the error rate

Constant value in the volume law

Exponentially decreasing in the area law

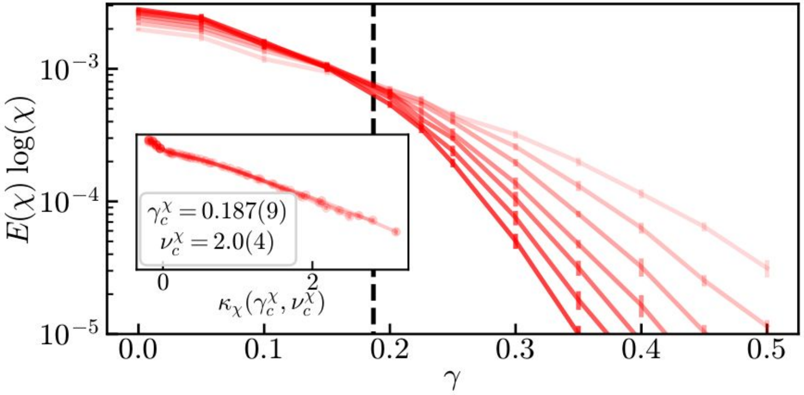


Time-average after saturation

We detected the MIPT by employing the “wrong”order of limit!

1. $t \to \infty$
2. $\chi \to \infty$

How do the microscopic details of a system impact the ensemble of MPS?

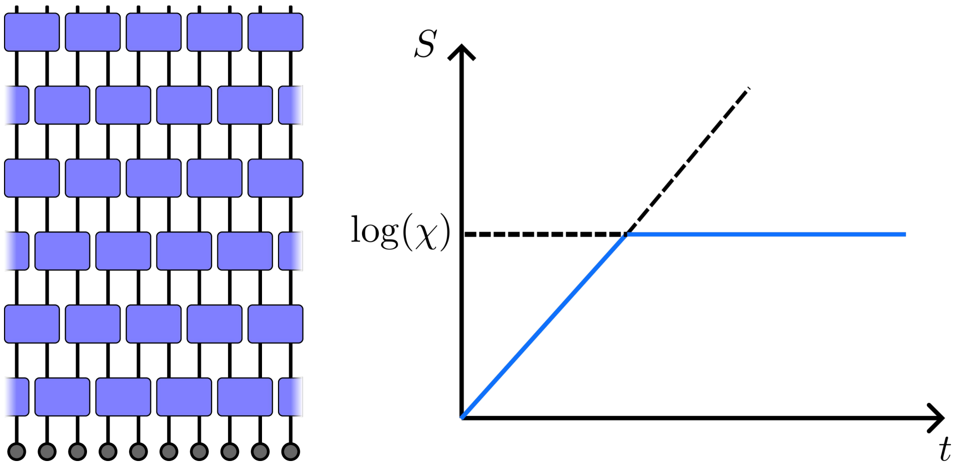


Measurement-induced phase transitions by matrix product states scaling, Guillaume Cecile, Hugo Lóio, Jacopo De Nardis, Phys. Rev. Research 6, 033220 (2024)

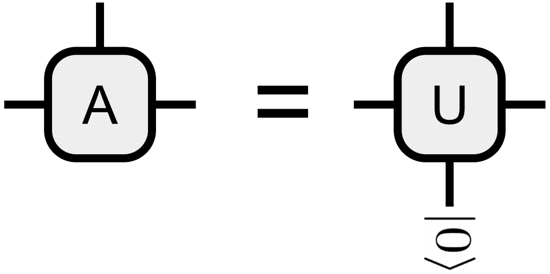
Characterizing correlations in ensembles of MPSs

A toy MPS ensemble for **ergodic** unitary evolution

Generic truncated evolution



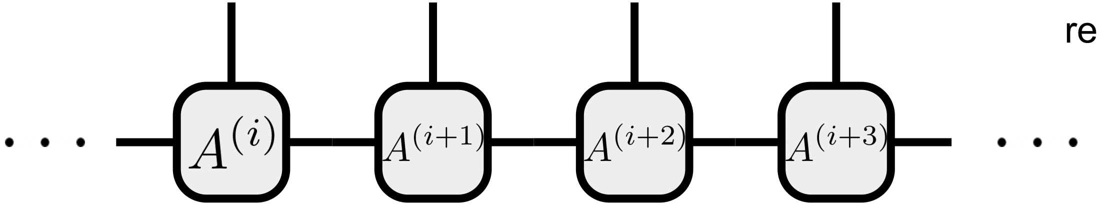
We can expect an ensemble of **random** matrix product tensors



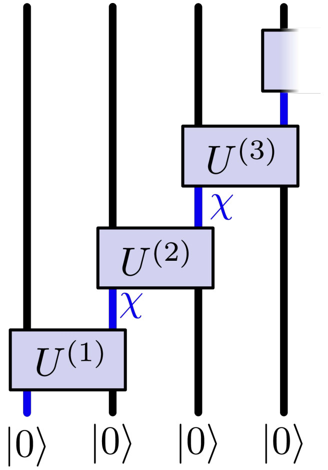
Haar random unitary matrix

U

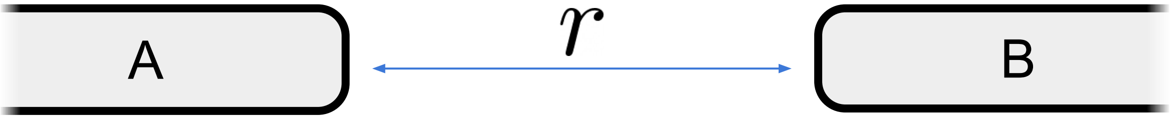
We can contract them to produce an ensemble of **random MPS (RMPS)**



Staircase representation



Mutual information as a probe for correlations



$$I_n(A:B) = S_A^n + S_B^n - S_{AB}^n$$

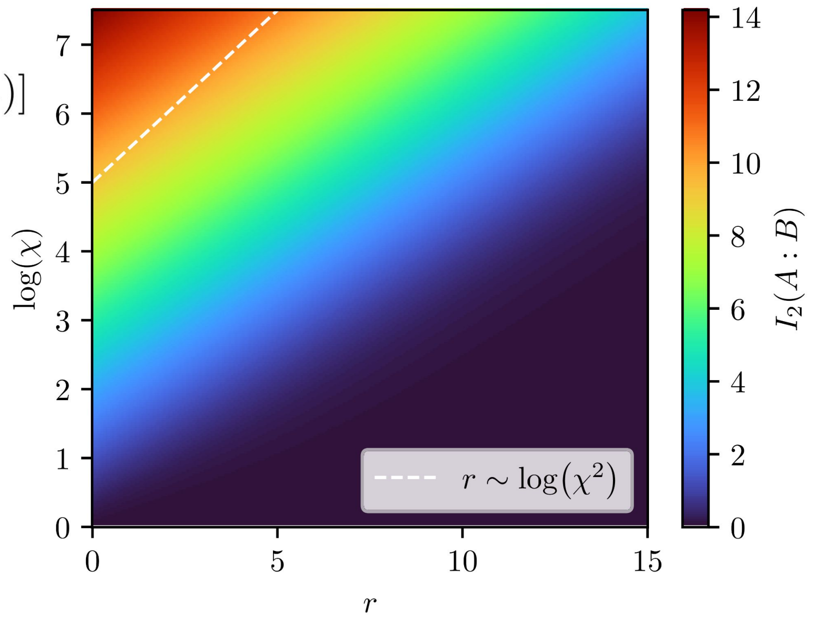
Averaging over RMPS

$$\mathbb{E}[I_2(A:B)] \simeq \log \mathbb{E}[\mathrm{Tr}(\rho_{AB}^2)] - \log \mathbb{E}[\mathrm{Tr}(\rho_A^2)] - \log \mathbb{E}[\mathrm{Tr}(\rho_B^2)]$$

$$\simeq \log\left(1 + \lambda_2^r \underbrace{\frac{4\chi^4 - 5\chi^2 + 1}{9\chi^2}}_{\mathcal{O}(\chi^2)}\right) \qquad \lambda_2 = \underbrace{\frac{2(\chi^2 - 1)}{4\chi^2 - 1}}_{\mathcal{O}(1)}$$

The mutual information stays finite over a distance

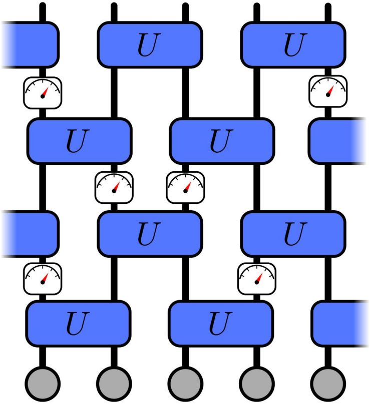
$$r \sim \log(\chi^2)$$



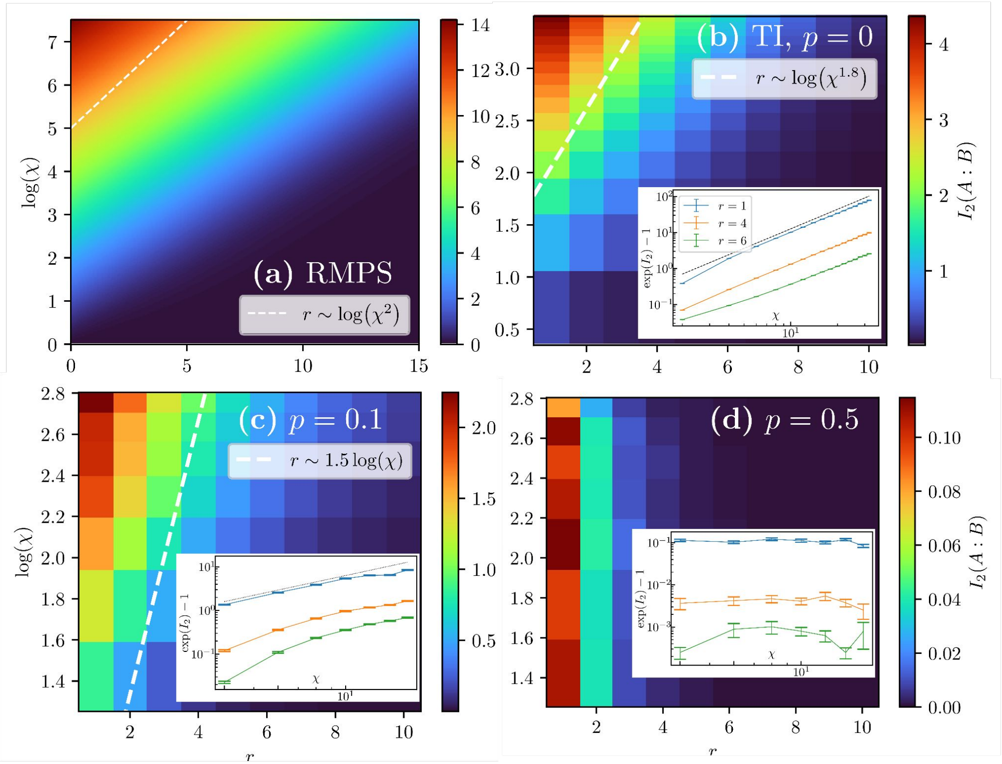
The Weingarten Calculus, Benoit Collins, Sho Matsumoto, Jonathan Novak, arXiv:2109.14890 (2022)

What about generic **monitored** circuits?

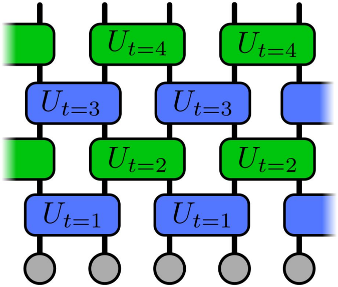
Unitary evolution with Haar random gates
+
Measurements with probability p



Mutual information



Translation invariant (infinite) Haar circuit



Conclusions

1. Ensembles of MPS retain properties of the microscopic evolution of systems even at **finite bond dimension**
2. Successfully probed MIPTs as a classical simulatability transition from the **error rate** of the **TDVP** method
3. Studied ensembles of MPSs that flow to different entanglement regimes
    a. Found a **logarithmic spreading** of correlations with bond dimension for generic volume-law MPSs (with a system-dependent prefactor)
    b. **Correlations do not spread** for area-law states, such as in the disentangling phase of the MIPT

In **collaboration** with

Jacopo De Nardis

Guglielmo Lami

Sarang Gopalakrishnan

Guillaume Cecile

Thank you!

Backup

Why are MIPTs **interesting**?

- Monitoring in quantum trajectories can describe the evolution of **open quantum systems** by unravelling the Linbladian
- Connection to **quantum error correction** and quantum channel capacity
- A replica trick approach to random hybrid circuits can map the MIPT to a ground state problem in an **effective spin model** (universality of dynamical phase transitions)
- The MIPT can generally be viewed as a **classical simulatability** transition

Matrix Product States (MPS)

The **experimental** challenge

Finding identical quantum trajectories for tomography is exponentially unlikely in circuit depth

Post-selection problem

Measurement-induced quantum phases realized in a trapped-ion quantum computer, Crystal Noel et al. Nature Physics volume 18, pages 760–764 (2022)

Experimental Realization of a Measurement-Induced Entanglement Phase Transition on a Superconducting Quantum Processor, Jin Ming Koh et al. arXiv:2203.04338 (2022)

Small number of qubits

Charge-sharpening transition in the XXX chain

Local charge **variance**

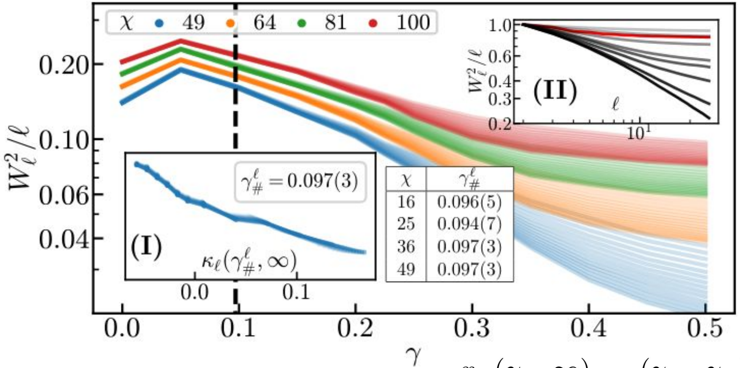
$$Q_\ell = \sum_{j\in\ell} S_j^z$$

$$W_\ell^2 = \langle Q_\ell^2\rangle - \langle Q_\ell\rangle^2 = \sum_{i,j\in\ell}\langle S_i^z S_j^z\rangle^c$$

$\gamma < \gamma_\#$ → $W_\ell^2 \sim \ell$

$\gamma > \gamma_\#$ → W_ℓ^2 sublinear

$$\gamma_\# < \gamma_c$$



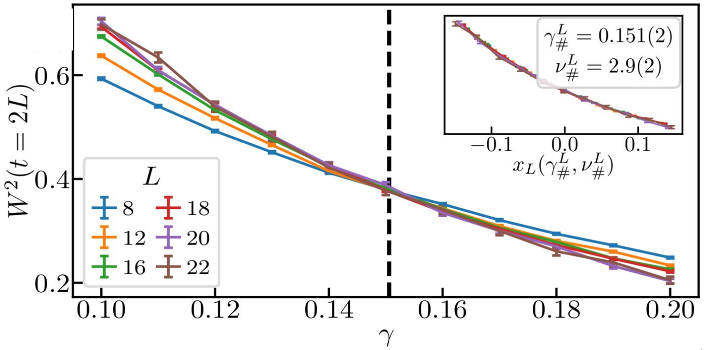
$x_A(\gamma_\mathcal{O},\infty) = (\gamma-\gamma_\mathcal{O})\log(A)$

Start with a superposition of all charge sectors

$$|\Psi(0)\rangle = \bigotimes_{i=1}^{L}(|\uparrow\rangle + |\downarrow\rangle)$$

Total charge **variance**

$$W^2(t) = \langle Q^2(t)\rangle - \langle Q(t)\rangle^2$$



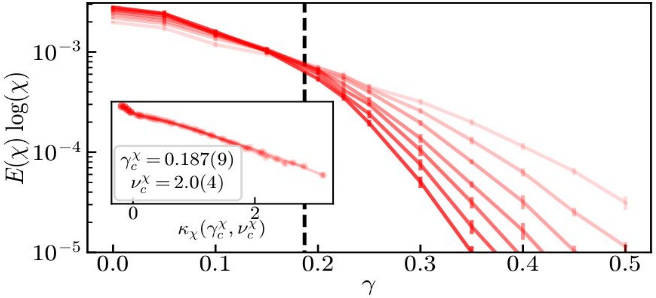
$x_A(\gamma_\mathcal{O},\nu_\mathcal{O}) = (\gamma-\gamma_\mathcal{O})A^{1/\nu_\mathcal{O}}$

Superlinear
to
sublinear
charge-sharpening
timescale transition

Comparison with **exact diagonalization**

Trotter decomposition of **unitary** evolution

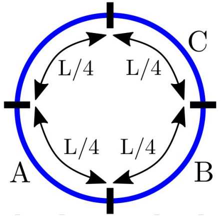
$$\hat{U}_{\mathrm{XXX}} \approx \prod_i \hat{u}^{\mathrm{XXX}}_{2i,2i+1} \prod_i \hat{u}^{\mathrm{XXX}}_{2i-1,2i} \qquad \hat{u}^{\mathrm{XXX}}_{i,i+1} = \exp\left[-i\delta t\left(\hat{S}^x_i \hat{S}^x_{i+1} + \hat{S}^y_i \hat{S}^y_{i+1} + \hat{S}^z_i \hat{S}^z_{i+1}\right)\right]$$



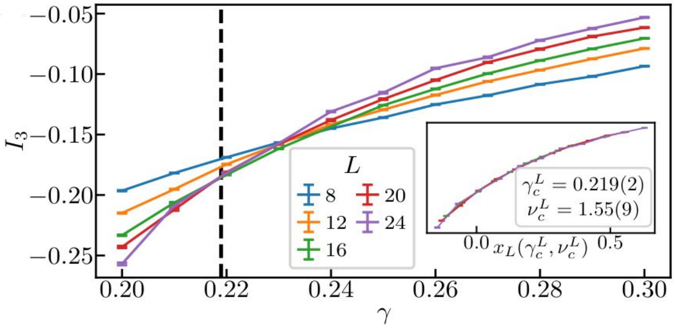
Projective Z measurements for the **monitoring** $P_\pm = \left(\frac{\mathbb{1}}{2} \pm \hat{S}^z_i\right)$

$|\psi\rangle \to P_\pm |\psi\rangle$ with probability $p_\pm = \langle\psi| P_\pm |\psi\rangle$

Tripartite mutual information



$$\mathcal{I}_{3,n}(A,B,C) \equiv S_n(A) + S_n(B) + S_n(C) - S_n(A\cup B)$$
$$- \; S_n(A\cup C) - S_n(B\cup C) + S_n(A\cup B\cup C)$$



$x_A(\gamma_{\mathcal{O}}, \nu_{\mathcal{O}}) = (\gamma - \gamma_{\mathcal{O}})A^{1/\nu_{\mathcal{O}}}$

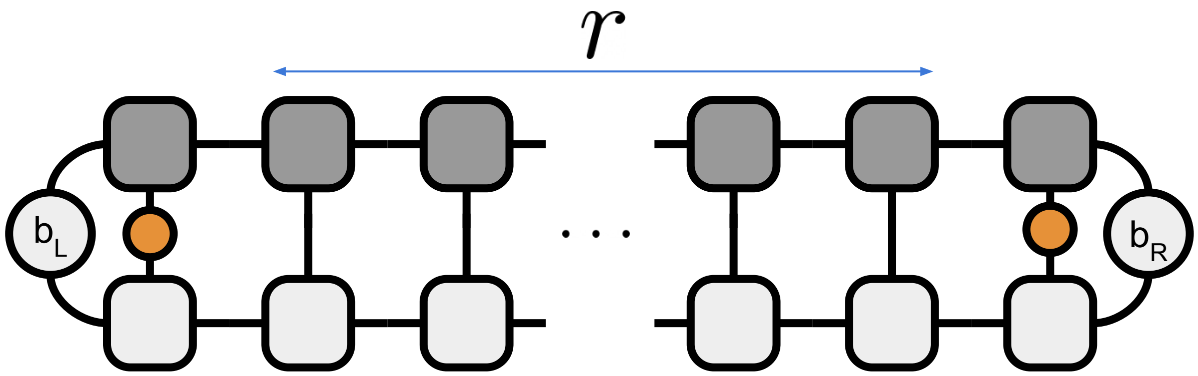
Expect γ_c^L to **drift left** in the thermodynamic limit

Predict $\gamma_c \simeq 0.21$

Measurement-induced transitions beyond Gaussianity: A single particle description,
Luca Lumia, Emanuele Tirrito, Rosario Fazio, and Mario Collura, PRR 023176 (2024)

How do MPS capture **correlations**?

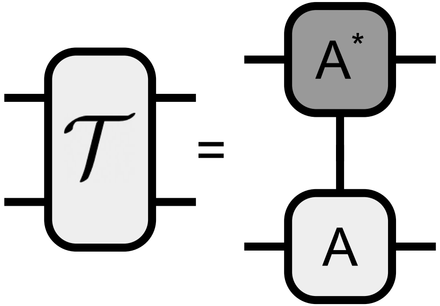
$$\langle\psi|\,\hat{O}_i\hat{O}_j\,|\psi\rangle =$$



2-point correlation function

If translational invariant…

Transfer matrix



Spectral decomposition

$$\mathcal{T} = \sum_k \lambda_k \,|r_k\rangle\,\langle l_k|$$

Normalization

$$\lambda_1 = 1$$

$$|r_1\rangle\,\langle l_1| + \lambda_2^r\,|r_2\rangle\,\langle l_2| + \dots$$

$$\langle\hat{O}_i\rangle\langle\hat{O}_j\rangle$$

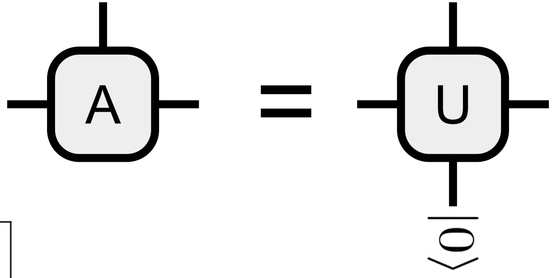
Disconnected

$$e^{-\frac{r}{\xi}} = \lambda_2^r \Leftrightarrow \xi = -\frac{1}{\log|\lambda_2|}$$

Correlation length

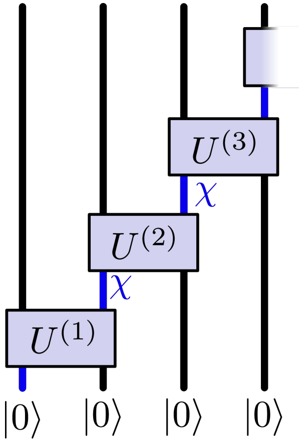
A toy model for **ergodic** unitary evolution

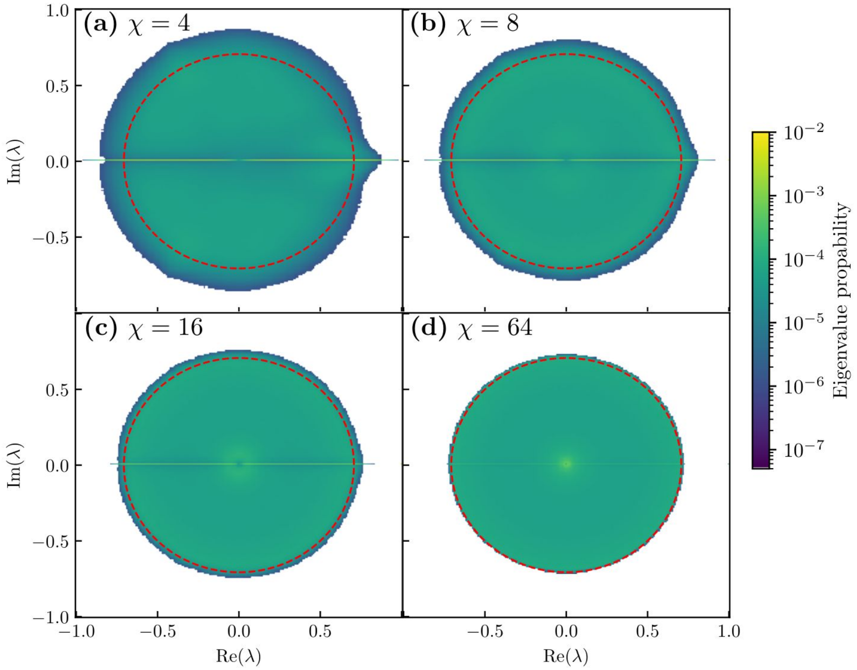
We expect ergodic unitary dynamics to produce a **random MPS (RMPS)**

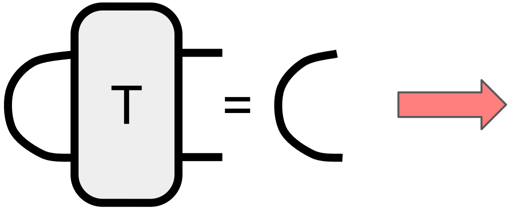


Haar Random unitary

U







$$\lambda_1 = 1$$

Random quantum channel

$$|\lambda_2| = 1 - \text{gap} \qquad \text{gap} = 1 - \frac{1}{\sqrt{d}}$$

Generating random quantum channels, Ryszard Kukulski, Ion Nechita, Łukasz Pawela, Zbigniew Puchała, Karol Życzkowski, J. Math. Phys. 62, 062201 (2021)

$\chi \to \infty$

- Volume-law entangled
- Finite correlation length

?