Learning quantum states and unitaries of bounded gate complexity

Laura Lewis

Joint work with Haimeng Zhao, Ishaan Kannan, Yihui Quek, Hsin-Yuan Huang, and Matthias C. Caro

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PRX Quantum, 2024

Motivation

Tomography is a fundamental task in quantum information and physics.

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What about quantum states/unitaries of bounded gate complexity?

Can we relate the complexity of learning quantum states/unitaries to that of creating them?

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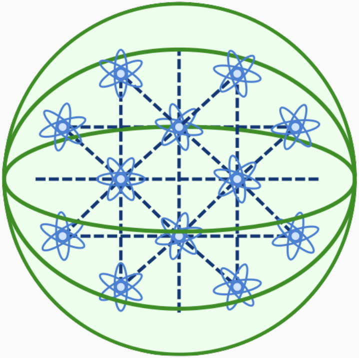
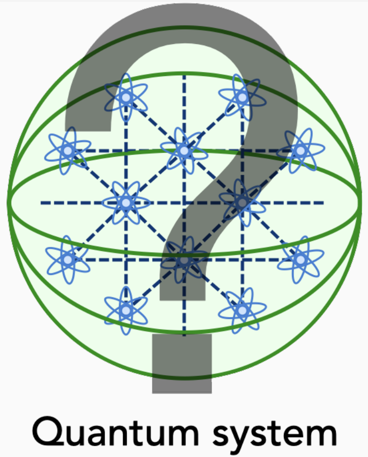
Quantum State Tomography



Quantum system

Quantum State Tomography



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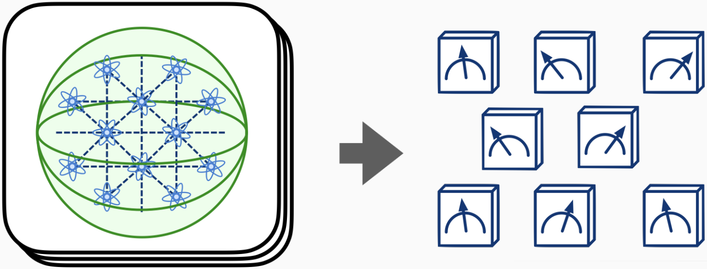
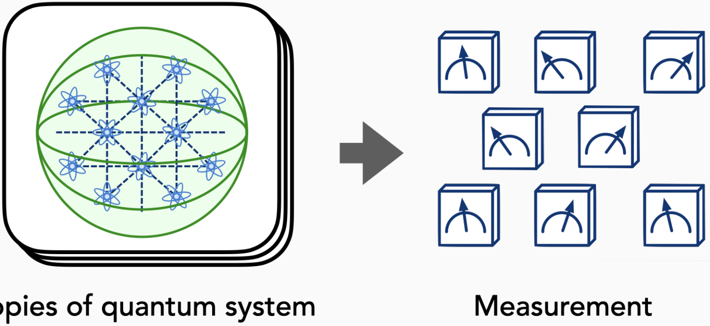


Copies of quantum system

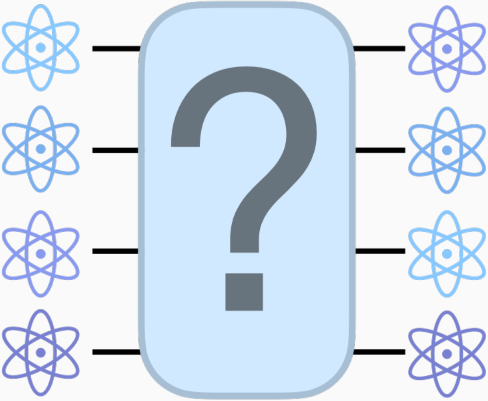
Measurement



Given N samples of an n-qubit quantum state ρ, learn $\hat{\rho}$ such that

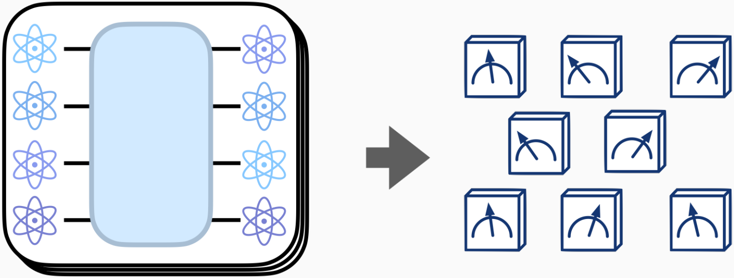
$$\mathrm{d}_{\mathrm{tr}}(\hat{\rho}, \rho) = \frac{1}{2}\|\hat{\rho} - \rho\|_1 < \epsilon.$$

Quantum Process Tomography



Quantum circuit

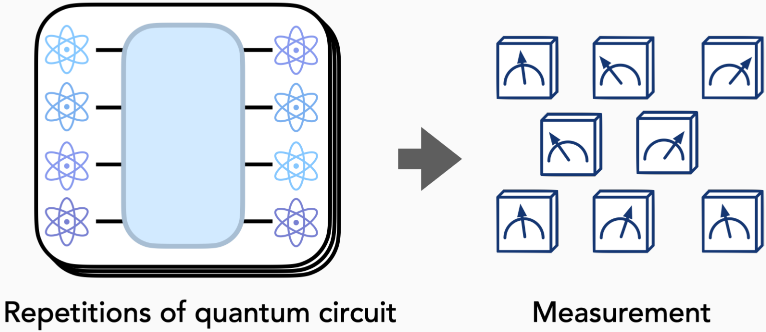
Quantum Process Tomography



Repetitions of quantum circuit

Measurement

Quantum Process Tomography



Given N queries to an n-qubit unitary U, learn $\hat{U}$ such that

$$\mathrm{d}_{\diamond}(\hat{U}, U) = \max_{\rho} \left\| (\hat{U} \otimes I)\rho(\hat{U} \otimes I)^{\dagger} - (U \otimes I)\rho(U \otimes I)^{\dagger} \right\|_1 < \epsilon.$$

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We want to minimize the *sample complexity*, i.e., the number N of copies of ρ or queries to U.

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We can also consider the *computational complexity*, i.e., the runtime of an algorithm.

Previous Work

Both quantum state and process tomography are known to require a sample complexity of $\Theta(4^n)$ in general[2].

[2] [Haah et al. 2017], [O'Donnell and Wright, 2016], [Haah et al. 2023]
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But physical quantum states/unitaries have bounded gate complexity[3].

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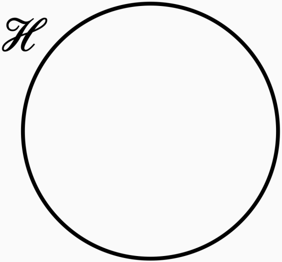

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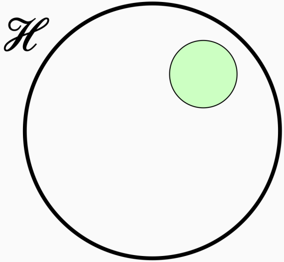

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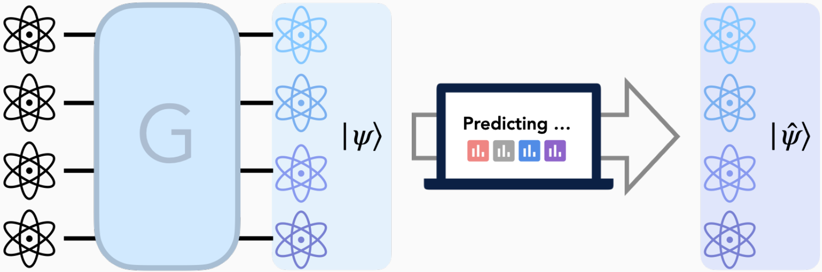
Learning States of Bounded Gate Complexity



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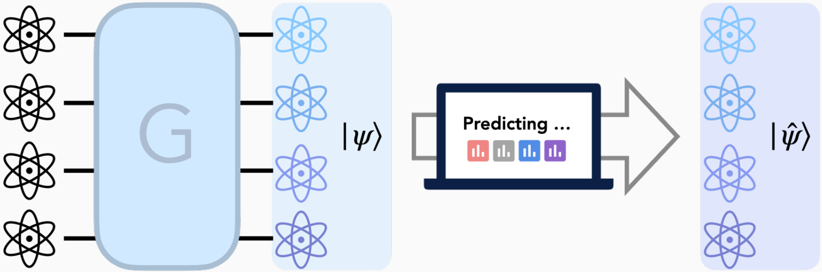

Given N copies of an n-qubit quantum state $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle = U|0\rangle^{\otimes n}$ where U consists of G gates, learn $\hat{\rho}$ such that

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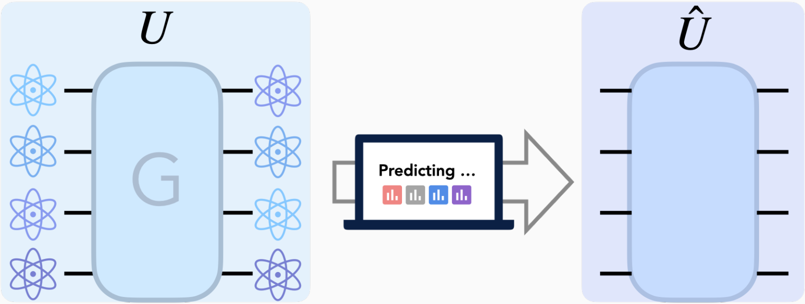
Learning Unitaries of Bounded Gate Complexity



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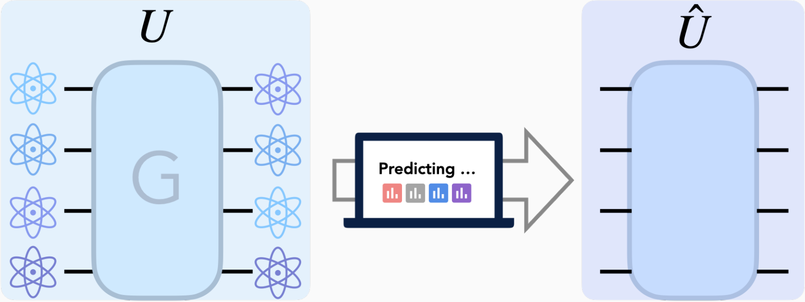

Given N queries to an n-qubit unitary U consisting of G gates, learn $\hat{U}$ such that

$$\mathrm{d}_\diamond(\hat{U}, U) = \max_\rho \left\| (\hat{U} \otimes I)\rho(\hat{U} \otimes I)^\dagger - (U \otimes I)\rho(U \otimes I)^\dagger \right\|_1 < \epsilon.$$

Outline

Main Results

Sample Complexity Upper Bounds

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Theorem (State learning)

The number of samples necessary and sufficient to learn an n-qubit quantum pure state with circuit complexity G within ϵ trace distance whhp is

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Previously, only an upper bound of $\tilde{\mathcal{O}}(nG^2/\epsilon^4)$ was known[6].

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Sample Complexity for Unitary Learning (Worst-Case)

Theorem (Worst-case unitary learning)

Any quantum algorithm learning an n-qubit unitary with circuit complexity G in diamond distance whp must use at least

$$\Omega(2^{\min\{G/(2C), n/2\}}/\epsilon)$$

queries. Meanwhile, there exists an algorithm using

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Thus, we can't hope for the same scaling as the state case for this distance metric.

Sample Complexity for Unitary Learning (Average-Case)

Instead, we turn to an average-case distance metric

$$\mathrm{d}_{\mathrm{avg}}(U, V) = \sqrt{\mathop{\mathbb{E}}_{|\psi\rangle\sim\mu}[\mathrm{d}_{\mathrm{tr}}(U|\psi\rangle, V|\psi\rangle)^2]}.$$

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Theorem (Average-case unitary learning)

There exists an algorithm learning an n-qubit unitary with circuit complexity G in root mean squared trace distance whp using

$$N = \tilde{\mathcal{O}}\left(G \min\left\{\frac{1}{\epsilon^2}, \frac{\sqrt{2^n}}{\epsilon}\right\}\right)$$

queries. Meanwhile, at least

$$\Omega(G/\epsilon)$$

queries are necessary.

Computational Hardness

Meanwhile, we prove strong computational limitations on learning even simple states/unitaries.

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Theorem (Computational hardness)

Any quantum algorithm that learns an n-qubit state/unitary with circuit complexity G to within ϵ trace distance/root mean squared trace distance requires

$$\exp(\Omega(\min(G, n)))$$

time, assuming the quantum sub-exponential hardness of RingLWE. Meanwhile, for $G = \mathcal{O}(\log n)$, an efficient learning algorithm exists.

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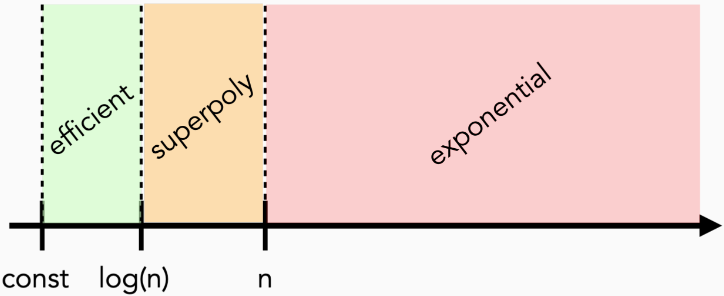

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To obtain the $\tilde{\mathcal{O}}(G/\epsilon^2)$ upper bound, we construct a covering net $\mathcal{N}$ over states with circuit complexity G.

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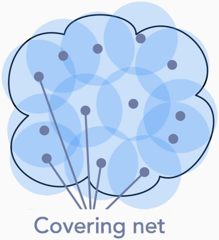

We show that

$$|\mathcal{N}| \leq e^{\tilde{\mathcal{O}}(G)}.$$

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Then, we use (modified) quantum hypothesis selection[7] to find a good state.

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The worst-case distance upper bound then follows by taking $\epsilon/\sqrt{2^n}$ since

$$\mathrm{d}_\diamond(U, V) \leq \sqrt{2^n}\mathrm{d}_{\mathrm{avg}}(U, V).$$

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Upper	$\tilde{\mathcal{O}}\left(G/\epsilon^2\right)$	$\tilde{\mathcal{O}}\left(G \min\left\{\frac{1}{\epsilon^2}, \frac{\sqrt{2^n}}{\epsilon}\right\}\right)$	$\tilde{\mathcal{O}}\left(2^n G/\epsilon\right)$
Lower	$\tilde{\Omega}\left(G/\epsilon^2\right)$	$\Omega\left(G/\epsilon\right)$	$\Omega\left(2^{\min\{G/(2C), n/2\}}/\epsilon\right)$

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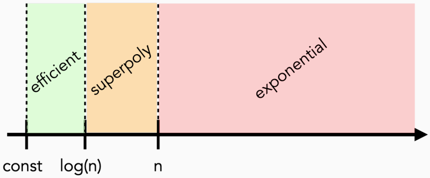
Lower	$\tilde{\Omega}\left(G/\epsilon^2\right)$	$\Omega\left(G/\epsilon\right)$	$\Omega\left(2^{\min\{G/(2C), n/2\}}/\epsilon\right)$

We also show a sharp transition in computational hardness at $G \sim \log n$.



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- Can the sample complexity results for unitary learning be made tight with respect to ϵ?
- Can we obtain better bounds for a fixed, known circuit structure?