Tracial embeddable strategies

Lifting MIP* tricks to MIPco

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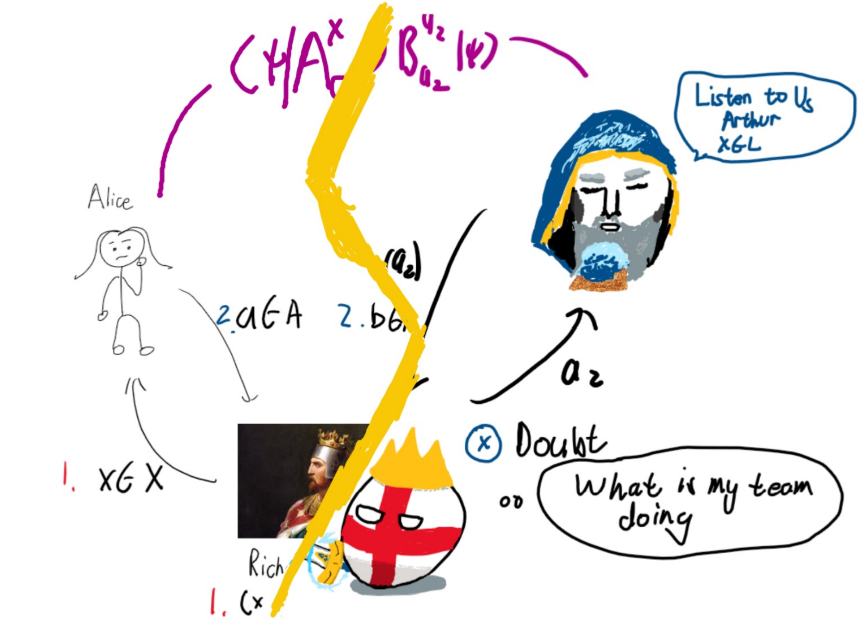
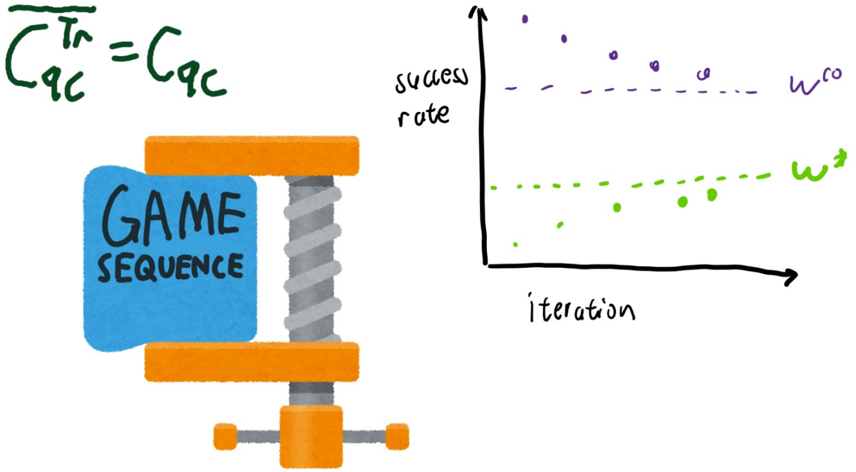




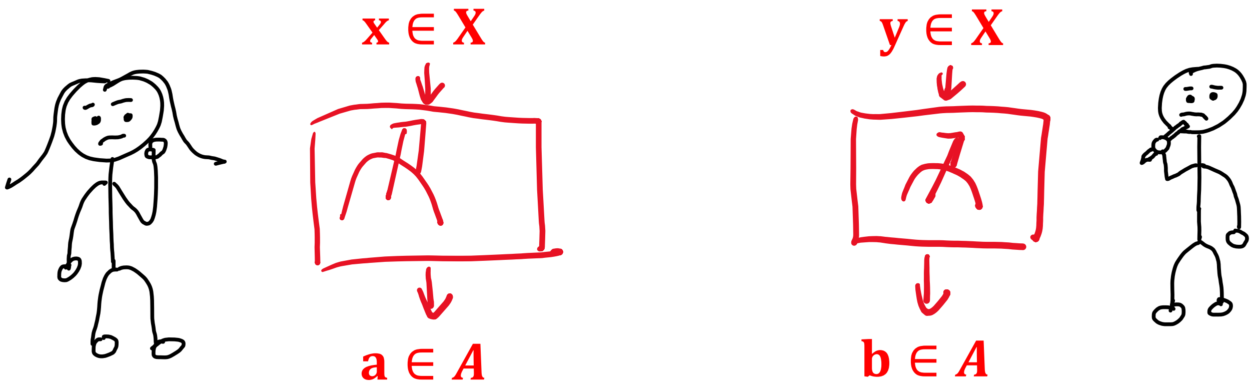
*We note that the "co" modifier on both sides of the equation MIPco=coRE refer to different things!

1. Quantum correlations and Models of entanglement.
2. Tracial embeddable strategies
3. Applications

1. Quantum correlations and Models of entanglement.

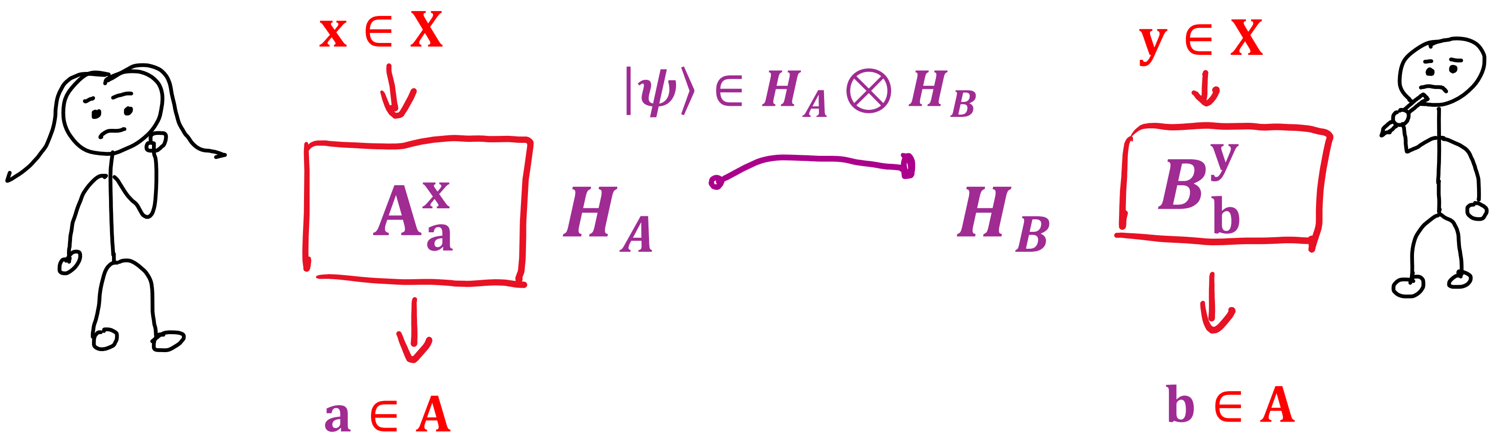
Correlation set

- **Mathematical model** design to describe a **physics experiment** between two **separated players.**



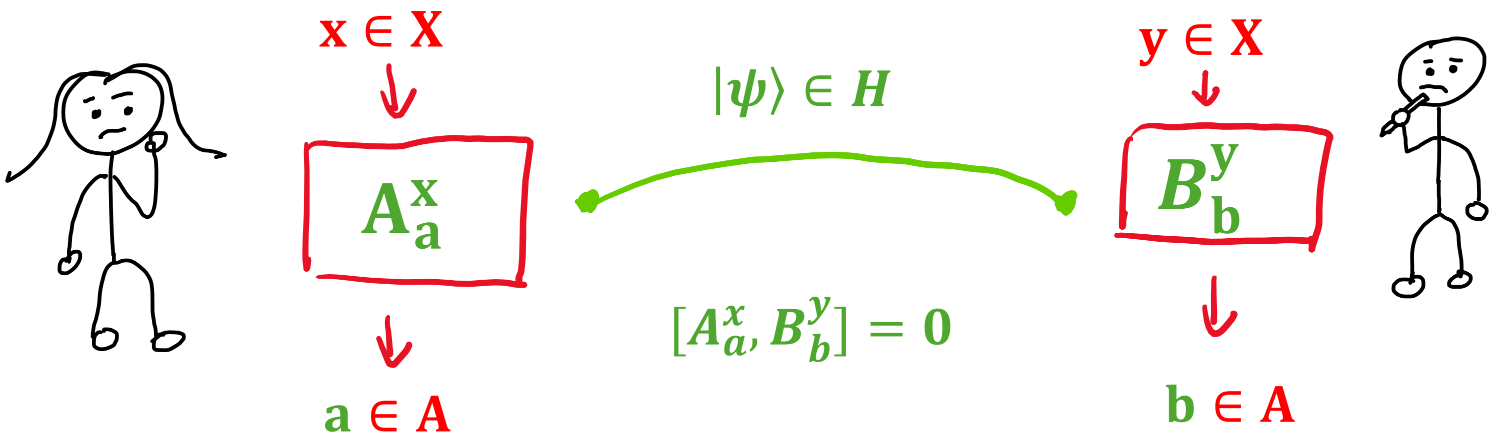
- **Correlations:** The joint probability distributions of output between the two parties (given the input). $P(a, b|x, y)$.
- Represented as vectors in $[0,1]^{|X|\times|X|\times|A|\times|A|}$

Quantum tensor correlations



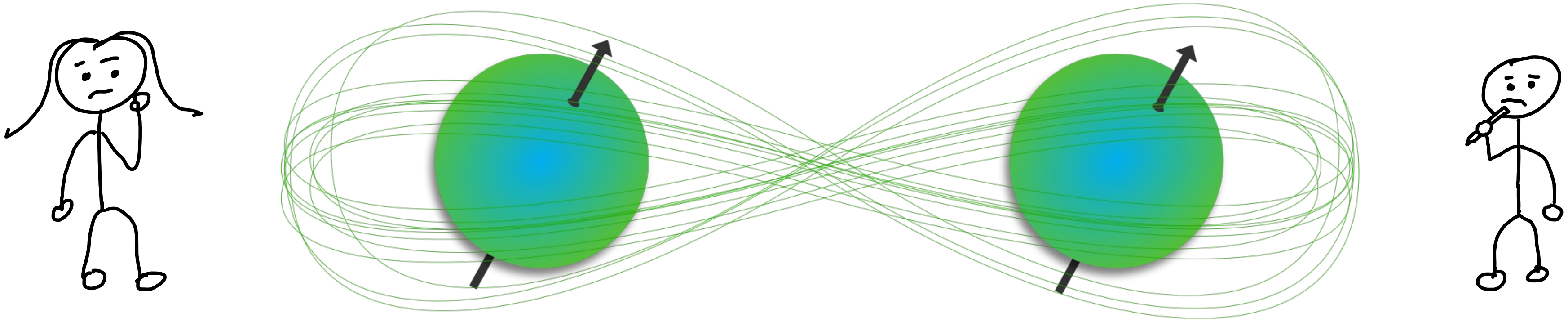
- $P(a, b|x, y)$ is a quantum **tensor** correlation if $P(a, b|x, y) = \langle\psi|A_a^x \otimes B_b^y|\psi\rangle$.
- $C_q(X, A)$: Set of quantum tensor correlations (subset of $[0,1]^{|X|\times|X|\times|A|\times|A|}$).

Quantum commuting correlations



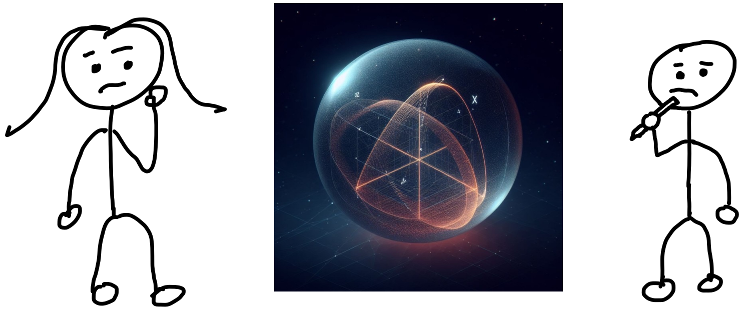
- $P(a,b|x,y)$ is a quantum **commuting** correlation if $P(a,b|x,y) = \langle\psi|A_a^x B_b^y|\psi\rangle$.
- $C_{qc}(X,A)$: Set of quantum commuting correlations.
- $C_q(X,A) \subsetneq C_{qc}(X,A)$ [MIP*=RE theorem], important implications for operator algebra, complexity/algorithm and physics.

Correlations



- Two parties in a different lab: quantum **tensor** correlation.

Correlations



- Different space: quantum **tensor** correlation.
- However, if the two parties are operating in the same system in the same space, the model in principle be in quantum **commuting** correlation (Tsirelson's problem).
- The different models of entanglement can be tested in theory similar to a bell test (with estimated $\sim 10^{20}$ questions and answer pairs).

Quantum strategies

- $P(a,b|x,y) \in C_{qc}(X,A)$
- POVMs $\{A_a^x\}, \{B_b^y\}$, and state $|\psi\rangle$ is a **quantum strategy** for P if

$$P(a,b|x,y) = \langle \psi | A_a^x B_b^y | \psi \rangle$$

- Strategy describes the behavior of the experimentors , correlation describes what the observer sees!

- Many quantum correlations have interesting theories behind strategies that achieve correlations that **are close** to the given correlations.
    - Robust self-testing (CHSH, tilted CHSH, Magic square).
    - Quantum soundness (Tensor code test).
- However, they are proven mostly in the tensor product model.
- Since the commuting operator model only differs in the **infinite-dimensional case**. These theorems are often hard to generalize.

Simple finite dimensional things that we enjoy

- The Hilbert Schmidt norm: $|X|_2 = \sqrt{\frac{1}{n} Tr(X)}$
- The reduced density matrix: $Tr_{H_A}(|\psi\rangle\langle\psi|) = \sigma$
- Observable switching trick: $(\sigma \otimes I)|\Phi\rangle = (I \otimes \sigma^T)|\Phi\rangle$

- Many of these simple things are not realizable in the infinite-dimensional case in general!

2. Tracial embeddable strategies

Tracial embeddable strategies

- A class of commuting operator strategies with many of the “nice” properties from finite-dimensional strategies.

Tracial embeddable strategies

- A commuting operator strategy $\{A_a^x\}, \{B_b^y\} \subseteq B(H), |\psi\rangle \in H$ is a **tracial embeddable strategy** if there exist a tracial von Neumann algebra (A, Φ) in **standard form** on the Hilbert space
  - $|\psi\rangle = \sigma|\Phi\rangle$ for some **positive** element σ $\in A$
  - $\{A_a^x\} \in A$
  - $\{B_b^y\} \in A'$, the commutant of A

Finite dimension example:

- For $P(a,b|x,y) \in C_q(X,A)$

In $M_n(\mathbb{C}) \otimes I$ (algebra)

$$P(a,b|x,y) = \langle \Phi | (\sqrt{\sigma} \otimes I)(A_a^x \otimes B_b^y)(\sqrt{\sigma} \otimes I) | \Phi \rangle$$

In $I \otimes M_n(\mathbb{C})$ (commtant)

Finite dimension example:

- For $P(a,b|x,y) \in C_q(X,A)$

Square root of the density matrix

Maximally entangled state

$$P(a,b|x,y) = \langle \Phi | (\sigma \otimes I)(A_a^x \otimes B_b^y)(\sigma \otimes I) | \Phi \rangle$$

$$|\psi\rangle = \sigma |\Phi\rangle$$

- This can always be achieved by enlarging one of the Hilbert spaces!

Main theorem

- We show that

$$\overline{C_{qc}^{Tr}(X,A)} = C_{qc}(X,A)$$

- $C_{qc}^{Tr}(G)$ - set of correlations generated by tracial embeddable strategies.
- This means that we can always assume this class of "nice" strategies if we do not require exact correlations when it comes to commuting operator correlations.

Proof of the main theorem


ARRETE!
C'EST ICI L'EMPIRE DE LA MORT

Proof of the main theorem

Idea: embedding a general von Neumann algebra into a tracial von Neumann algebra of the standard form + approximation of normal state.

- Main proof technique:
  - Connes-Tomita-Takesaki theorem (II_∞)
  - Generalization to Choi's proof that $C^*(F_d)$ is RFD (II_1).
- Approximation mainly comes from getting the underlying state to be normal + approximable by a positive element.

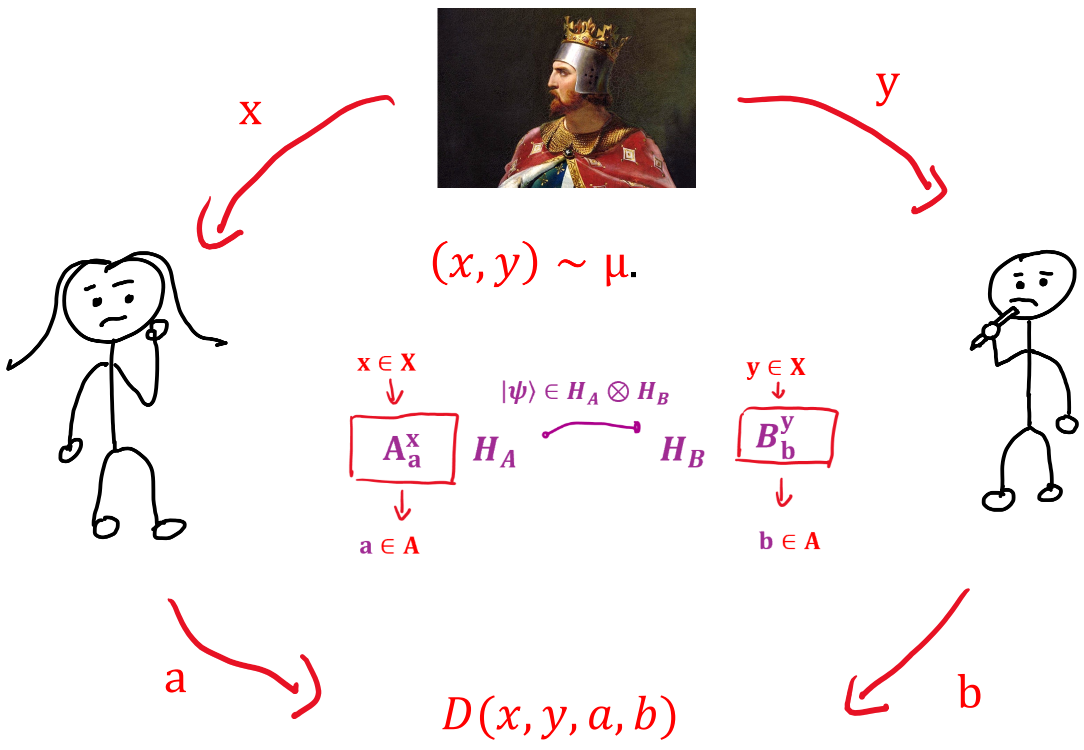
3. Applications

Main theme

- Tracial embeddable strategies have the structures of finite-dimensional tensor product strategies, many of the proof techniques can be lifted!
- Any property which does not require exact correlations for tracial embeddable strategies would hold for commuting operator correlations.
- This gives a framework to lift many theorems from finite-dimensional tensor product cases to the commuting operator correlations.

Nonlocal games

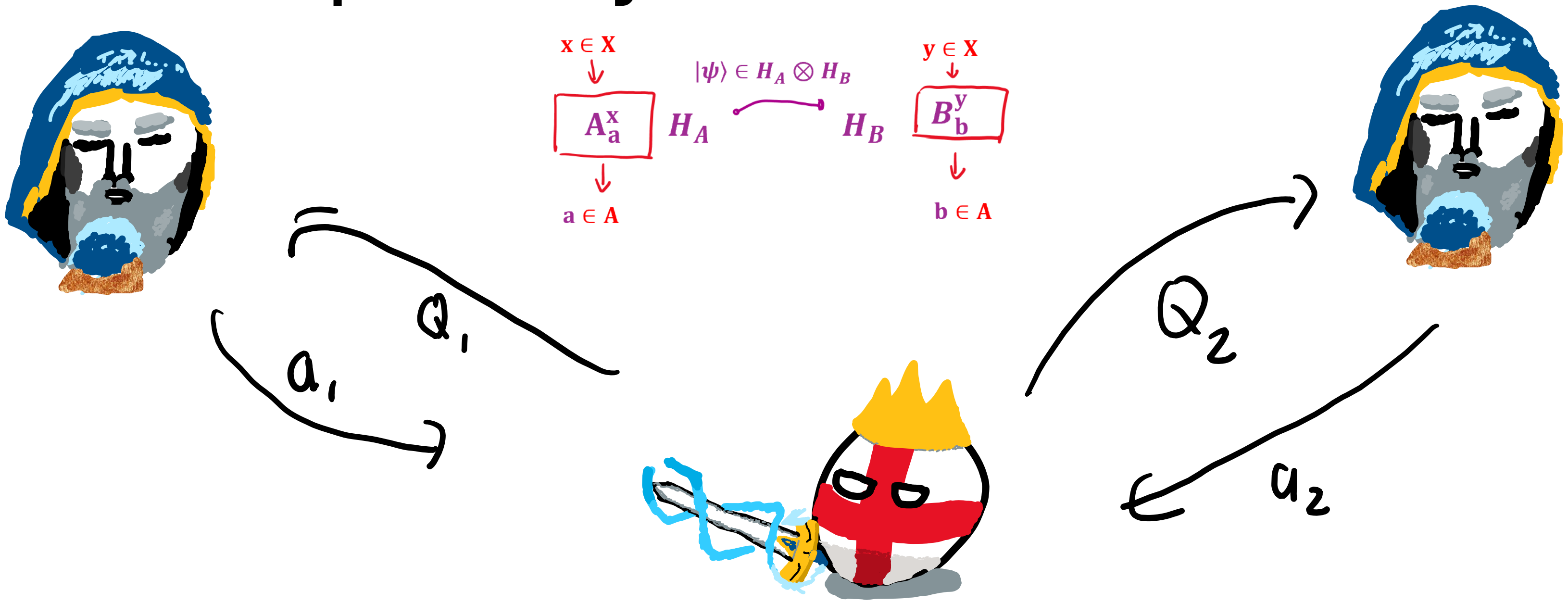
- $G = (\mu, D)$ is a **two-player nonlocal game** with question set Q and answer set A, where
  - μ distribution over $Q \times Q$ (**Question distribution**)
  - $D: Q \times Q \times A \times A \rightarrow \{0,1\}$ (**Decision predicate**)
- The referee samples $(x, y) \sim \mu$. Sends x to Alice and y to Bob.
- Without communicate with each other Alice responds with a, and Bob with b.
- Players win if $D(x, y, a, b) = 1$.
- The players can potentially employ some quantum correlations (tensor, commuting), which can help them win the game more (Example: CHSH).



Approximating nonlocal games

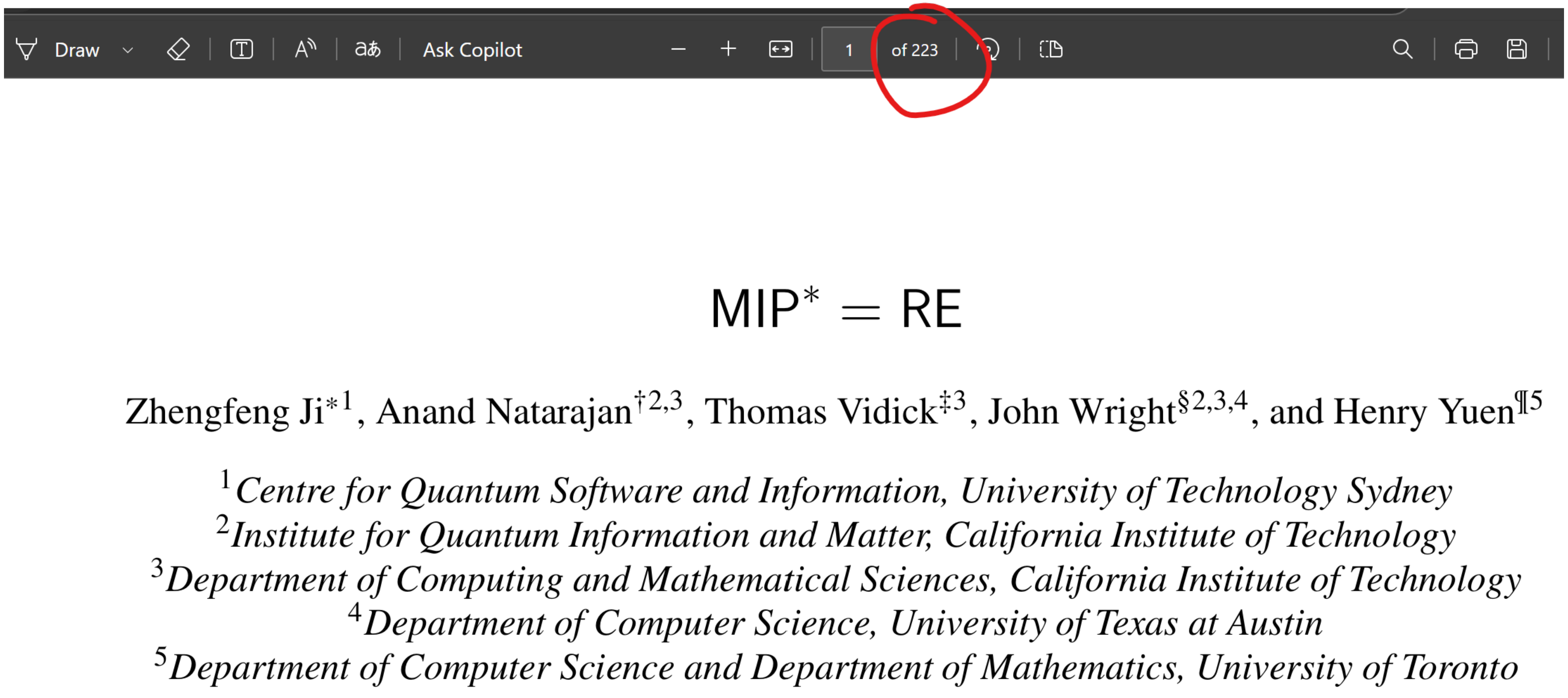
- If I give you $G = (\mu, D)$, how difficult is it to approximate the optimal success rate for a game (up to 0.5)?
- (Quantum tensor) This problem is known to be equivalent to the halting problem [$\mathrm{MIP}^* = \mathrm{RE}$], due to the connection with computer science.

Interactive proof system



- Nonlocal games are known as a Multiplayer Interactive Proof system (MIP) in the computer science language.
- MIP*: MIP, but the prover gets access to tensor product correlations.
- This model is equivalent to solving the halting problem (Impossible!).
- This theorem is proven using tools from both the CS community and the quantum information community.

MIP*=RE



MIP* = RE

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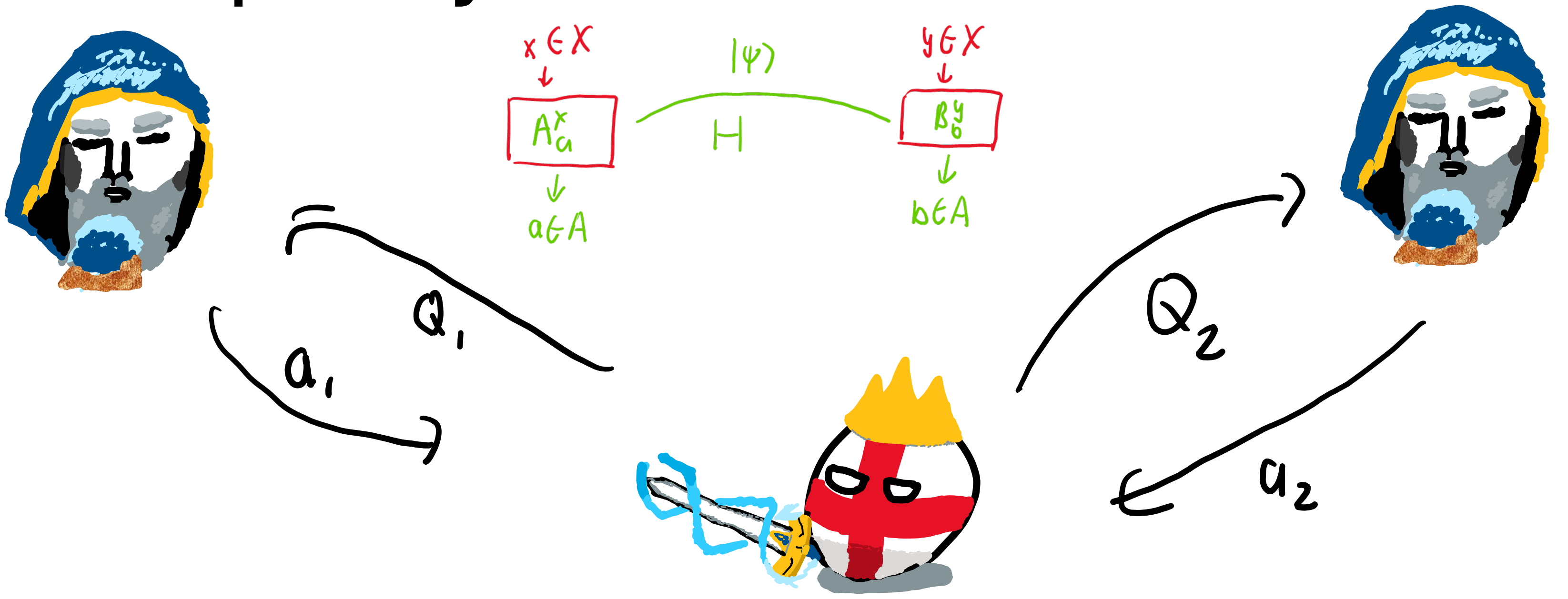
Approximating nonlocal games

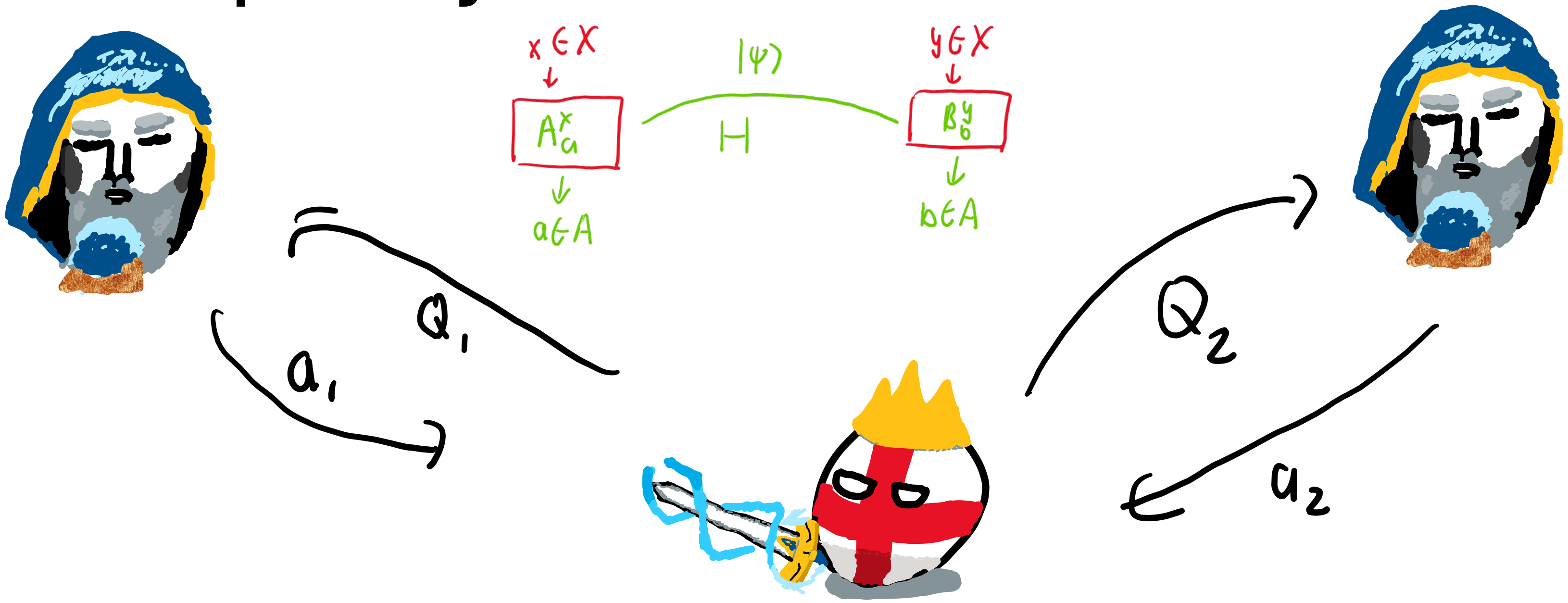
- If I give you $G = (\mu, D)$. How difficult is it to approximate the optimal success rate for a game (up to 0.5)?
- (Quantum tensor strategies) This problem is known to be equivalent to the halting problem [$MIP^* = RE$], due to the connection with computer science.
- (Quantum Commuting strategies) Unknown!

The complexity of MIP^{co}



- The complexity of approximating the optimal success rate under the commuting operator model for a game. Conjectured to be equivalent to the non-halting problem.
- Problem: Most of the theorems from Quantum information do not generalize in the Quantum Commuting model!

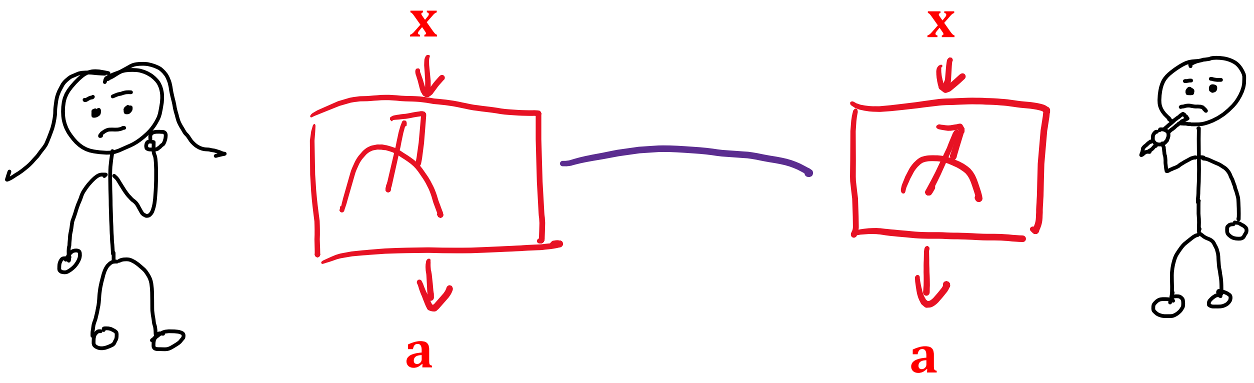
The complexity of $\mathrm{MIP}^{\mathrm{co}}$



- Due to the main theorem, this is equivalent to approximating the optimal success rate over tracial embeddable strategies.
- Using tracial embeddable strategies, many of the techniques MIP^* can be moved to the commuting operator model.

Rounding

- A correlation is **synchronous** if the output is the same when given the same input

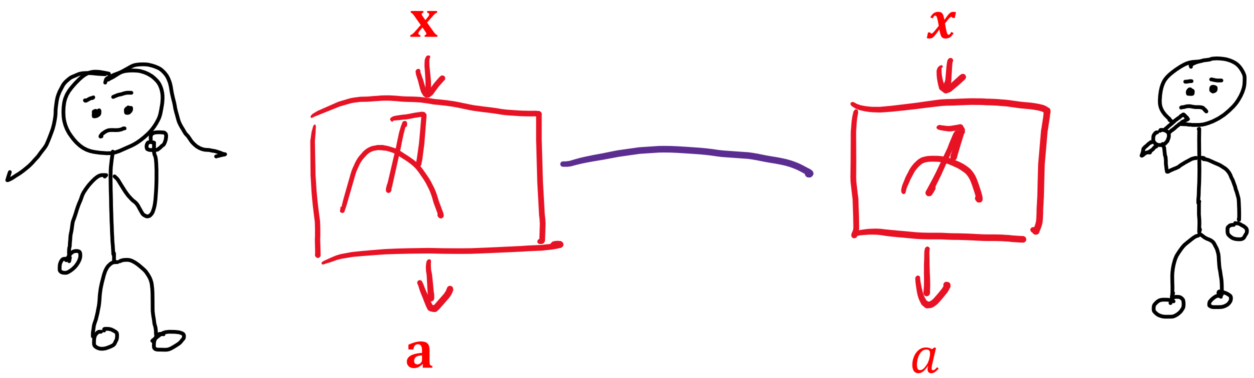


$P(a, a|x, x) = 1$

- Well-studied correlations [KPS17, HMP16+]
- Key part of the $\mathrm{MIP}^* = \mathrm{RE}$ theorem.

Rounding

- A correlation is $\boldsymbol{\delta}$**-almost synchronous** if the output is the same **most of the time** when given the same input



$$P(a, a|x, x) \geq \mathbf{1} - \boldsymbol{\delta}$$

- Rounding: Can all $\boldsymbol{\delta}$**-almost synchronous correlation** be approximated by **synchronous correlation**?

Rounding – Our result

- [Vid21, Pau21]: True in the tensor product case (Proof relies on the trace).
- We show the rounding theorem in the commuting operator case.
- By assuming the underlying strategy is **tracial embeddable**, we can repeat [Vid21]'s argument in the commuting operator model.
- Rounding + [JNV+22b]: Quantum low-degree is sound in the commuting operator model of entanglement.
  - Combining this with [BFL91] this implies that $\mathrm{MIP} = \mathrm{NEXP} \subseteq \mathrm{MIP}^{\mathrm{co}}$.

Robust self-test

- Self-testing (tensor): Correlations with unique strategy (up to local isometry).
- Self-testing (CO): Correlations realizable by a unique state on $C_A^X \otimes_{max} C_A^X$
- Robust self-testing (tensor): All **approximate** strategies are **close** (in $||\cdot||_2$) to the unique strategy.
- Robust self-testing (CO): **Unclear!**

Robust self-test – Our result

- We prove robust self-testing for tracial embeddable strategies for a version of the Pauli-basis test
- The proof relies on the state-dependent Gowers-Hatami theorem (tracial embeddable strategies).
- Our rigidity statement (roughly) looks like this:

If $\{A_a^x\}, \{B_B^y\}, |\psi\rangle$ is a tracial embeddable strategy which succeed at the Pauli basis test with probability $1-\epsilon$, then there exist two partial isometry V_A, V_B such that $V_A(V_B \otimes I) = V_B(V_A \otimes I)$ and

$\to V_A \otimes V_B$

$$||V_B(V_A \otimes I)\,|\psi\rangle - |Aux\rangle|EPR\rangle^{\otimes n}||^2 \leq O(poly(\epsilon))$$

Other applications – NPA Hierarchy.

- An algorithm design to upper bound the commuting operator model of entanglement.

maximize $\sum_{x,y,a,b} q(x,y)V(a,b|x,y)\langle\psi|A_a^x B_b^y|\psi\rangle = \mathrm{Tr}[C\ G^1] =: \omega_1$

subject to

(i) $\langle\psi|\psi\rangle = 1$

(ii) $\sum_a \langle\psi|A_a^x|\psi\rangle = 1, \qquad \sum_b \langle\psi|B_b^y|\psi\rangle = 1$

$\sum_a \langle\psi|A_a^x B_b^y|\psi\rangle = \langle\psi|B_b^y|\psi\rangle, \qquad \sum_b \langle\psi|A_a^x B_b^y|\psi\rangle = \langle\psi|A_a^x|\psi\rangle$

(iii) $\langle\psi|A_a^x A_{a'}^x|\psi\rangle = \delta_{aa'}\langle\psi|A_a^x|\psi\rangle$

(iv) $\langle\psi|A_a^x B_b^y|\psi\rangle = \langle\psi|B_b^y A_a^x|\psi\rangle$

(v) $G^1 \succeq 0$

SDP

$$\underbrace{\begin{pmatrix} \langle\psi|\psi\rangle & \langle\psi|A_{a_1}^{x_1}|\psi\rangle & \cdots & \langle\psi|B_{b_1}^{y_1}|\psi\rangle & \cdots \\ \langle\psi|A_{a_1}^{x_1}|\psi\rangle & \langle\psi|A_{a_1}^{x_1}A_{a_1}^{x_1}|\psi\rangle & \cdots & \langle\psi|A_{a_1}^{x_1}B_{b_1}^{y_1}|\psi\rangle & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle\psi|B_{b_1}^{y_1}|\psi\rangle & \langle\psi|B_{b_1}^{y_1}A_{a_1}^{x_1}|\psi\rangle & \cdots & \langle\psi|B_{b_1}^{y_1}B_{b_1}^{y_1}|\psi\rangle & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}}_{G^1}$$

Other applications – NPA Hierarchy.

- By using tracial embeddable strategies, one can incorporate states involving density matrix into the gram matrix within the NPA Hierarchy.
  - Example: $\langle \tau | \sigma\, A_b^x\, B_b^y \sigma | \tau \rangle = \langle \tau | \mathrm{B}_\mathrm{b}^\mathrm{y} \sigma^2 \mathrm{A}_\mathrm{b}^\mathrm{a} | \tau \rangle$ ($|\tau\rangle$ is the tracial vector state!)
- This gives a variant of the NPA Hierarchy which gives more variables (potentially faster convergence??).

Thank you for your attention.

- Tracial embeddable strategies are a class of commuting operator strategies with many similar properties to a finite-dimensional tensor product strategy.
- Tracial embeddable strategies generate a set of correlations that are dense in the commuting operator strategies.
- Using the above two statements, we created a framework to lift theorems from the tensor product model to the commuting operator model.

QR code to Arxiv link