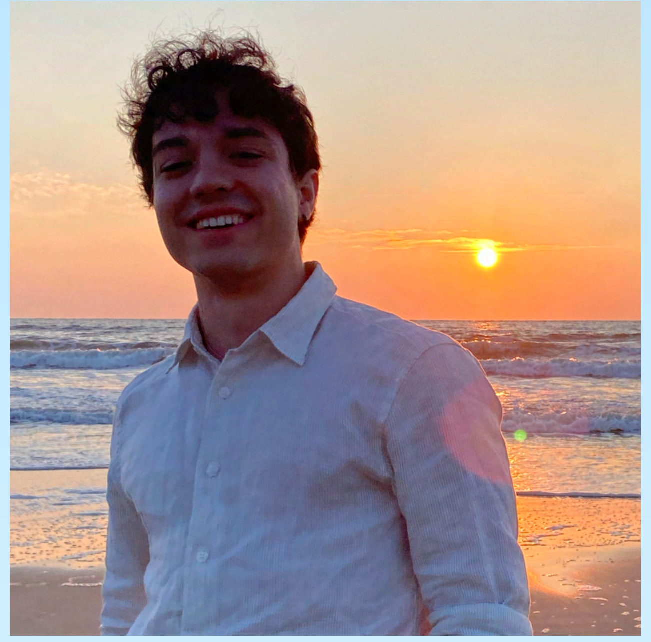
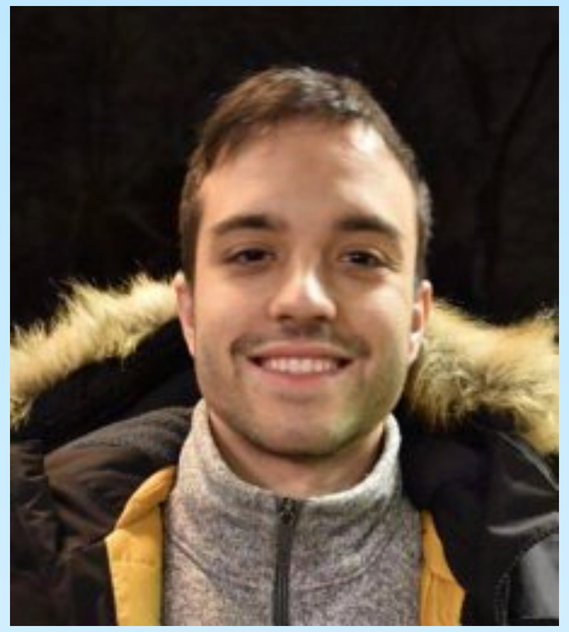
# Learning quantum states of continuous variable systems



Francesco A. Mele



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**Learning quantum states of continuous variable systems,** arXiv:2405.01431 (2024)

**Quantum Learning Theory** + **Continuous Variable Systems** ⟶ Our work

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↓ Extensively developed for **finite**-dimensional systems

↓ Dimension $= \infty$

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**Quantum state tomography** = Learning unknown quantum states

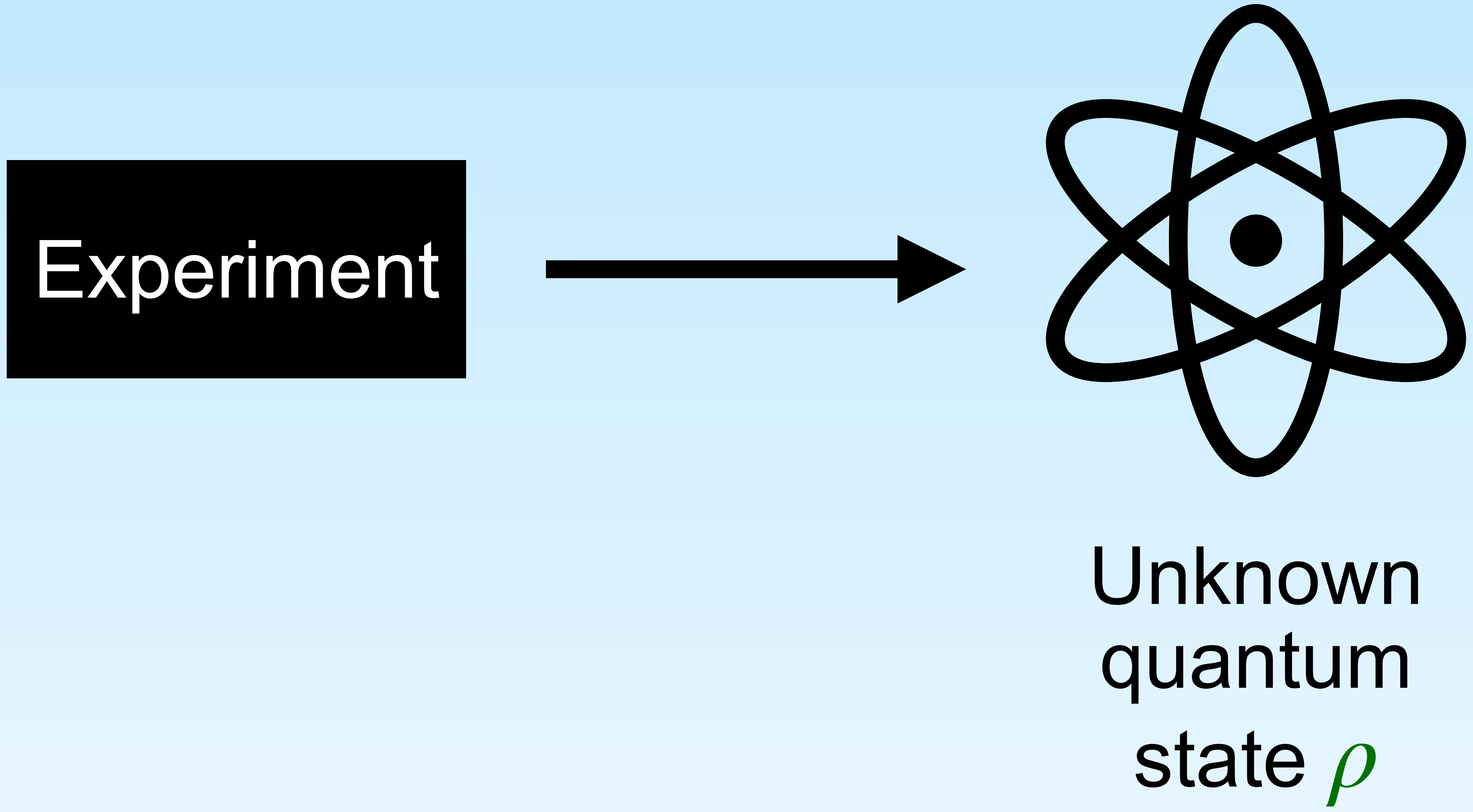
Despite many (heuristic) tomography methods in quantum optics,
the literature lacks "*rigorous performance guarantees*"

→ Our work fills this gap

# Outline

1. Quantum state tomography
2. CV systems
3. Quantum state tomography of CV systems:
   A. Energy-constrained states
   B. Gaussian states
   C. $t$-doped Gaussian states

# **Quantum state tomography** = Learning unknown quantum states

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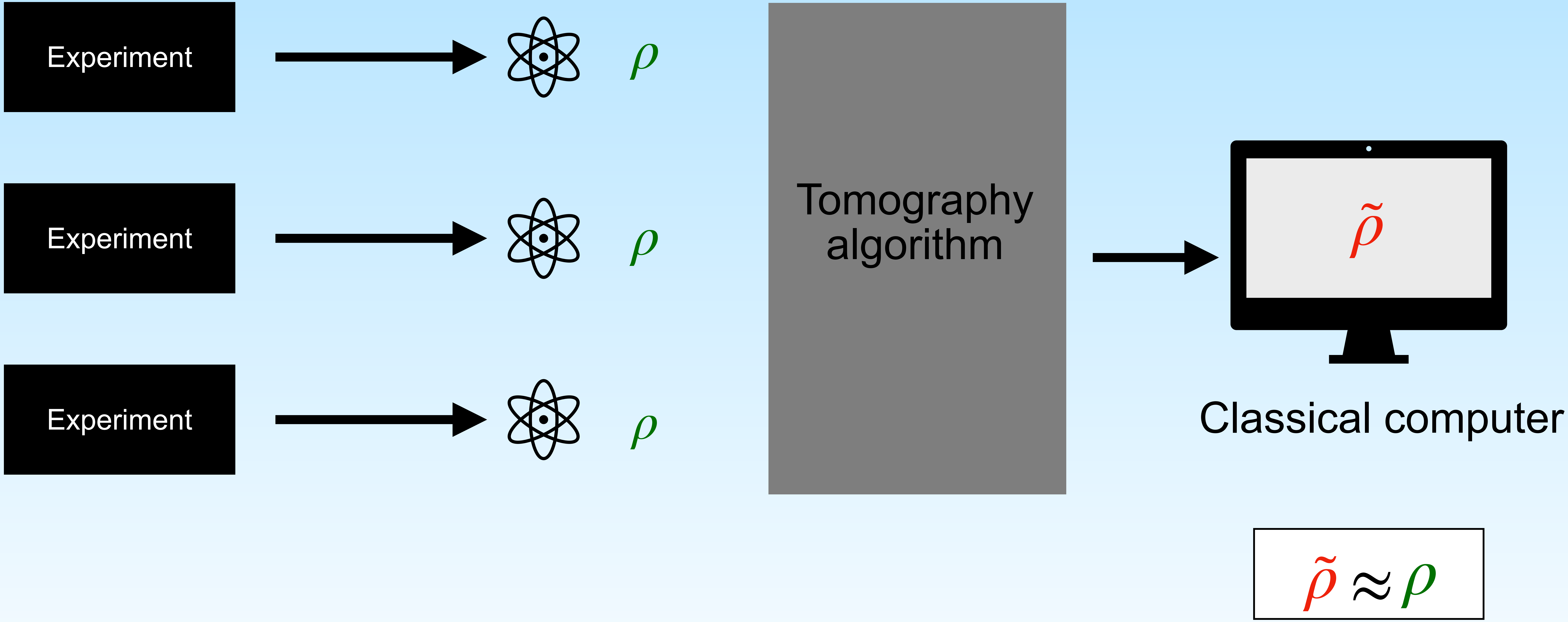
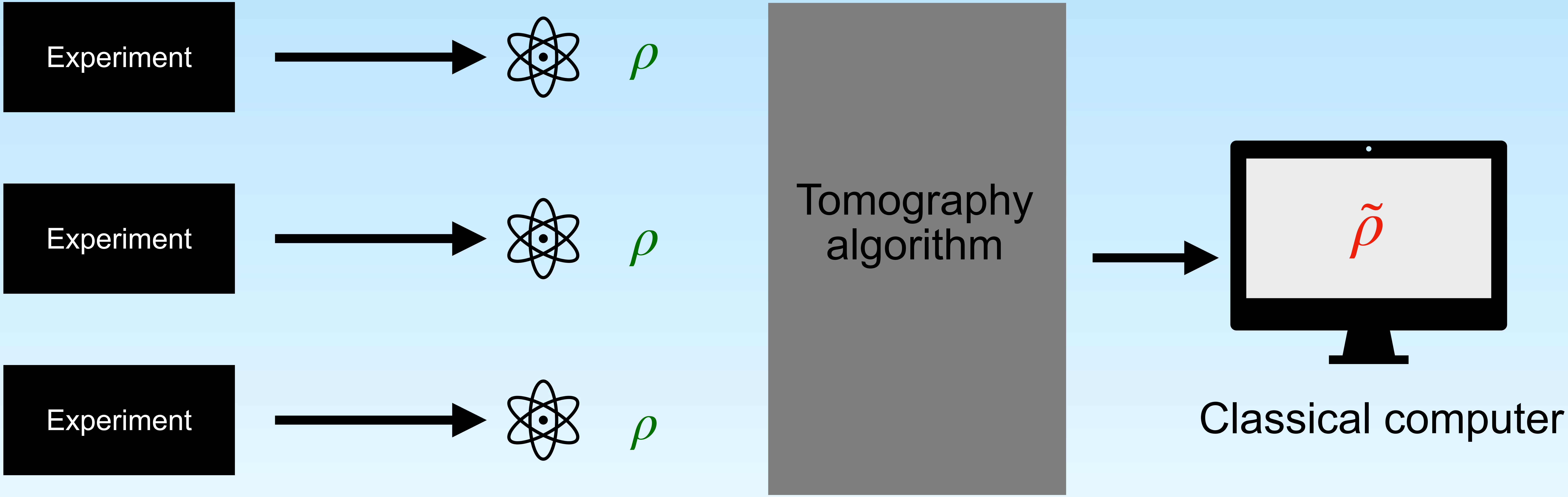
# Quantum state tomography



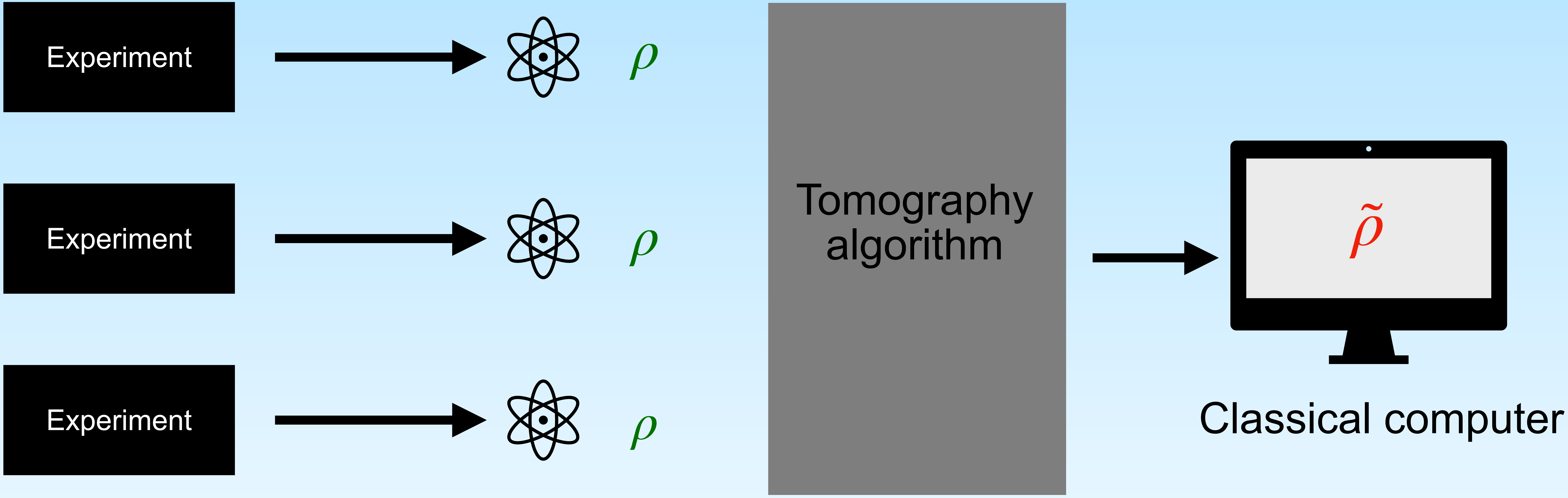
# Quantum state tomography




$$\tilde{\rho} \approx \rho$$

$$d_{\mathrm{tr}}(\tilde{\rho}, \rho) := \frac{1}{2} \|\tilde{\rho} - \rho\|_1$$

# Quantum state tomography




$$\Pr\left[d_{\mathrm{tr}}(\tilde{\rho},\rho) \leq \varepsilon\right] \geq 1-\delta$$

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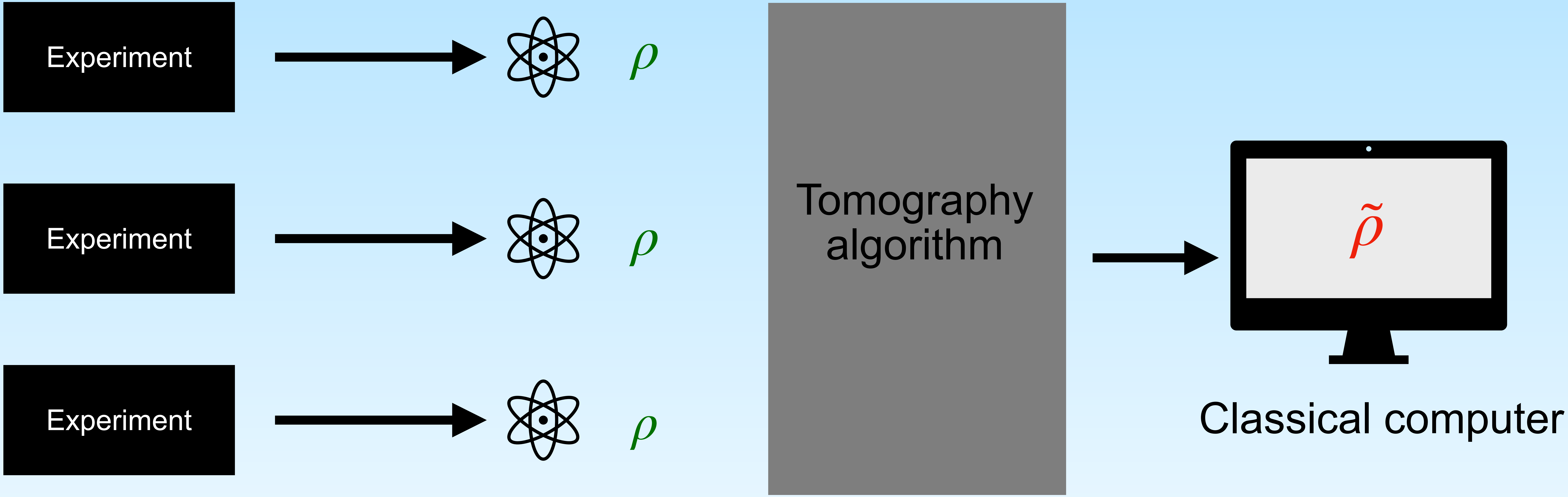
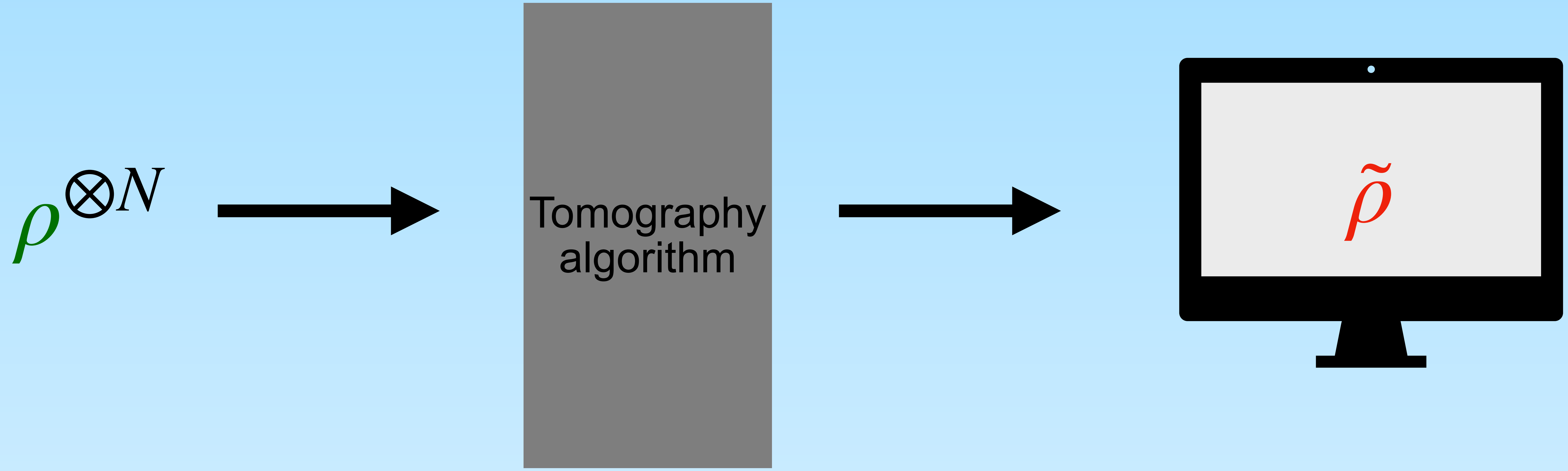
# Quantum state tomography




$$\Pr\left[d_{\mathrm{tr}}(\tilde{\rho},\rho) \leq \varepsilon\right] \geq 1-\delta$$

$\rho \in \mathcal{S}$ = some subset of the set of quantum states

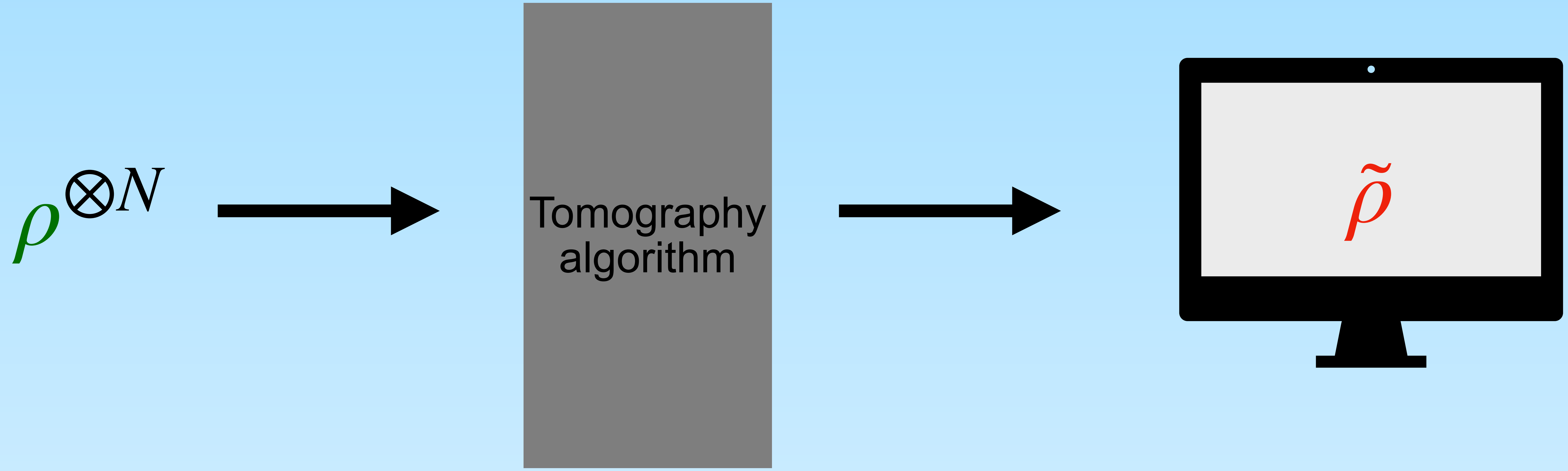
$$d_{\mathrm{tr}}(\tilde{\rho},\rho) := \frac{1}{2}\|\tilde{\rho}-\rho\|_1$$




**Problem 1** (Quantum state tomography)

Given $N$ copies of the (unknown) state $\rho \in \mathcal{S}$, the goal is to output $\tilde{\rho}$ such that

$$\Pr\left[d_{\mathrm{tr}}(\tilde{\rho},\rho) \leq \varepsilon\right] \geq 1-\delta$$

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**Definition**

The **sample complexity** $N(\mathcal{S}, \varepsilon, \delta)$ is the minimum $N$ satisfying **Problem 1**

**Example 1**

$$\mathcal{S} := \{\text{qu}D\text{it states}\} \longrightarrow N(\mathcal{S}, \varepsilon, \delta) = \tilde{\Theta}\left(\frac{D^2}{\varepsilon^2}\log\left(\frac{1}{\delta}\right)\right)$$

[R. O'Donnell and J. Wright, Efficient quantum tomography (2015)]
[J. Haah, A. W. Harrow, Z. Ji, X. Wu, and N. Yu, Sample-optimal tomography of quantum states (2017)]

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**Example** 2

$$\mathcal{S} := \{\text{qu}D\text{it pure states}\} \longrightarrow N(\mathcal{S}, \varepsilon, \delta) = \tilde{\Theta}\left(\frac{D}{\varepsilon^2}\right)$$

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$n$-qubit states ⟶ $D = 2^n$ ⟶ Tomography is inefficient

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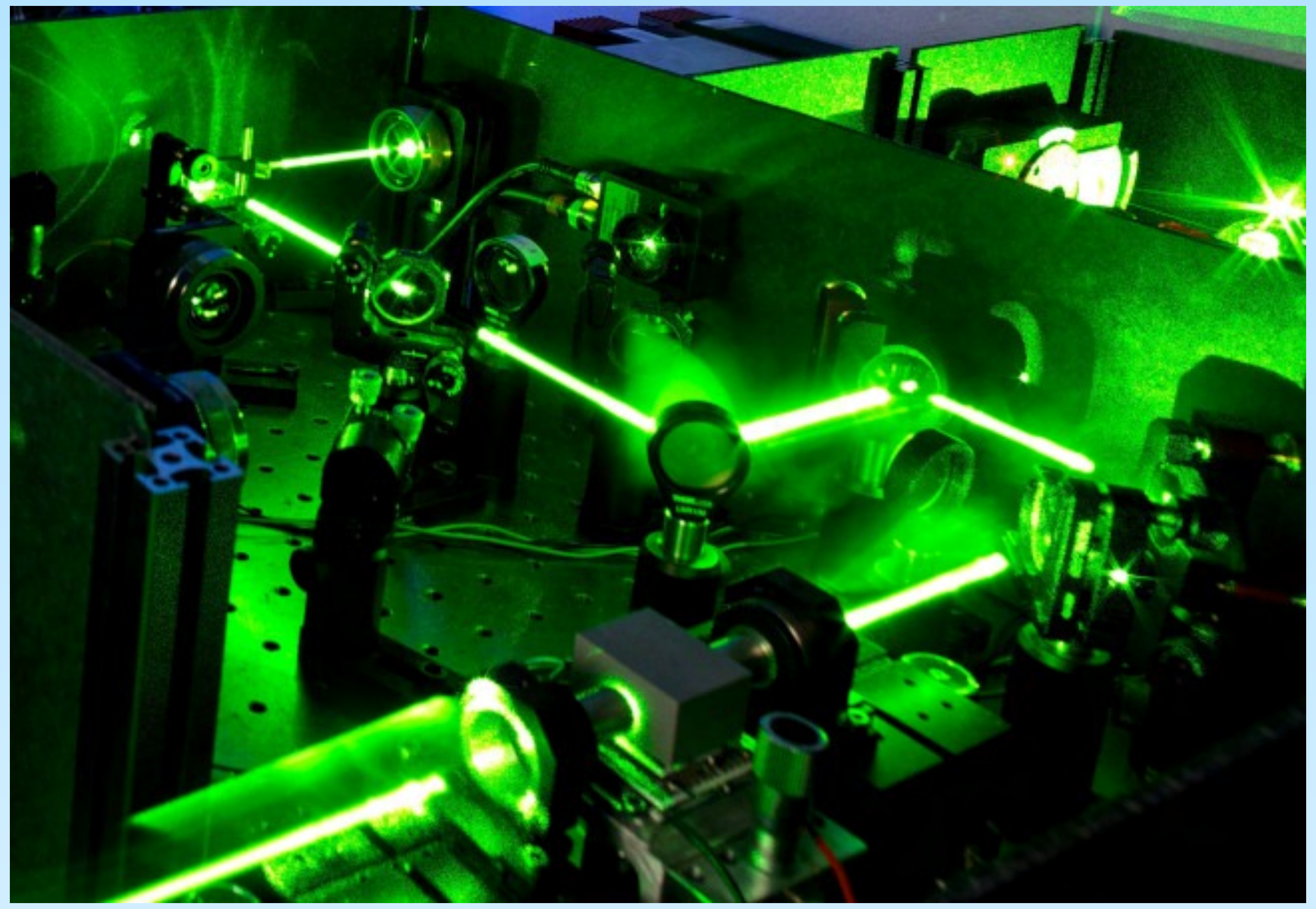
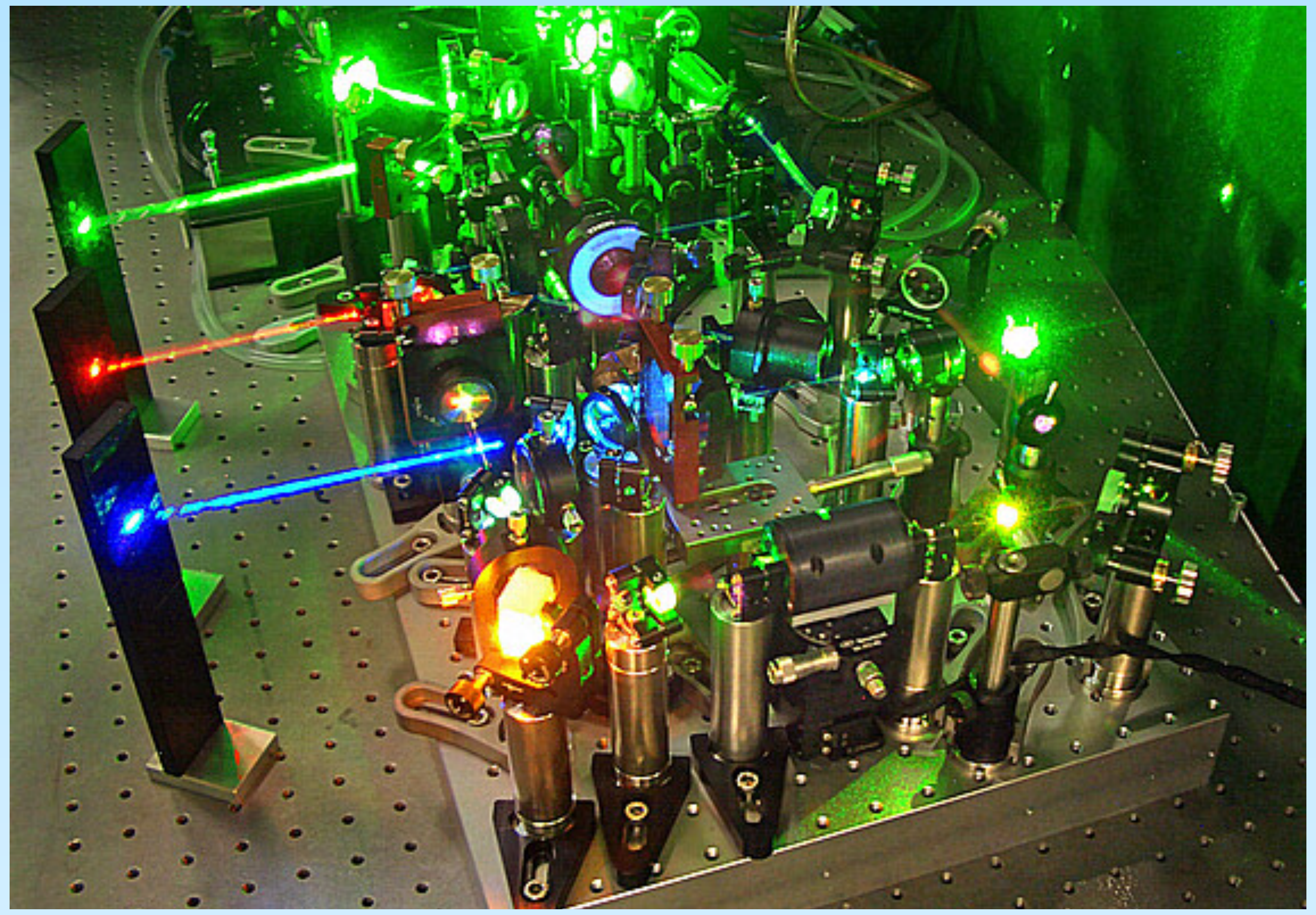
# CV systems

Quantum optical systems





1 ***mode*** ⟷ 1 qu$d$it with $d = \infty$

Fock states

Hilbert space $= \text{Span}\{\, |0\rangle \,, |1\rangle, \cdots \,, |d\rangle, |d+1\rangle, \cdots \}$

Vacuum state ($0$ photons)

($d$ photons)

A ***CV system*** consists in $n$ modes (i.e. $n$ qu$d$its with $d = \infty$)

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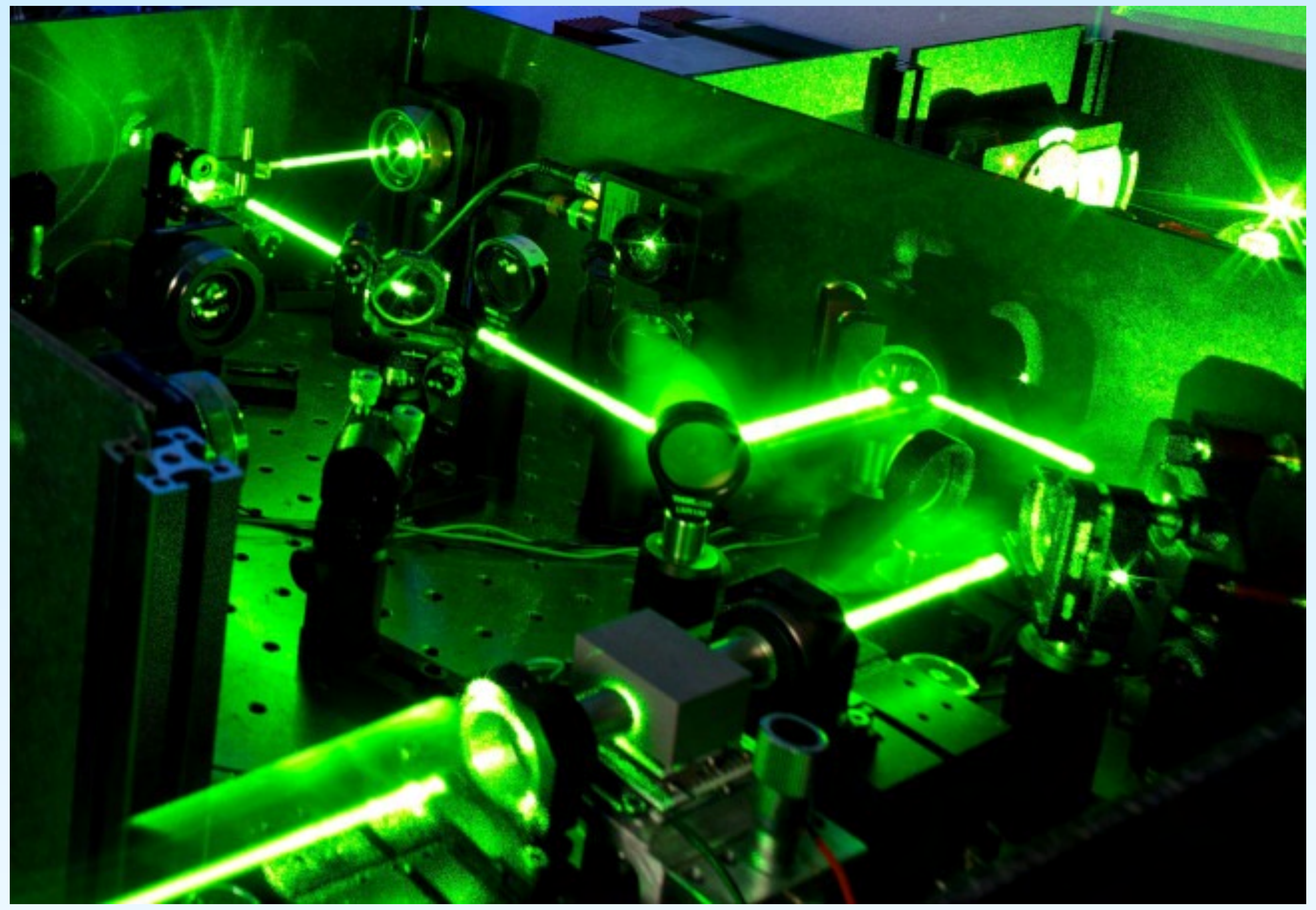

$\hat{E}$ = energy observable

$$\hat{E}\,|k_1\rangle \otimes |k_2\rangle \otimes \ldots \otimes |k_n\rangle = \left(\sum_{i=1}^{n} k_i\right)|k_1\rangle \otimes |k_2\rangle \otimes \ldots \otimes |k_n\rangle$$

Total number of photons

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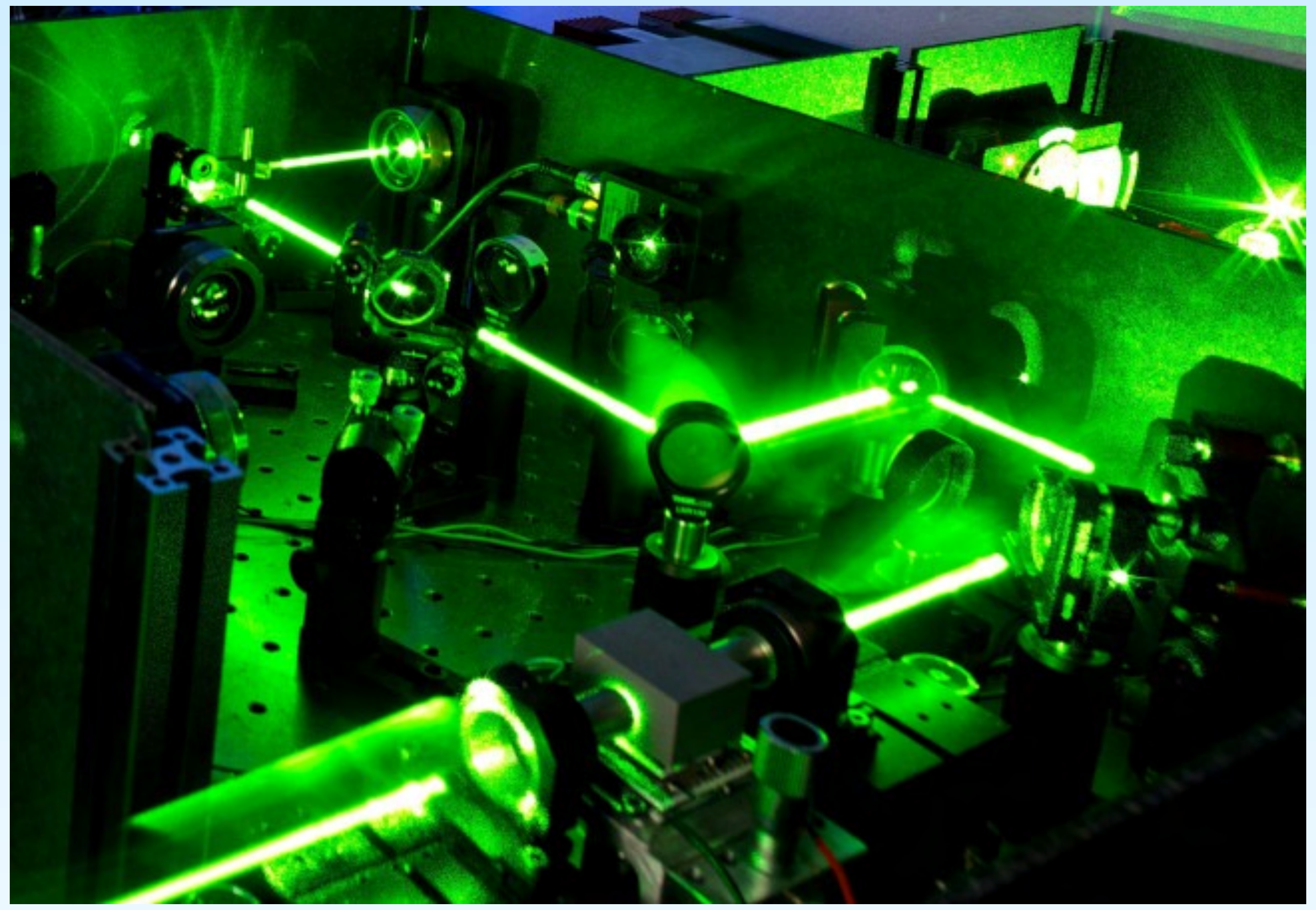

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**Assumption:**

The unknown state $\rho$ satisfies $\mathrm{Tr}\left[\rho\hat{E}\right] \leq E_{\mathrm{tot}}$ for some *known* $E_{\mathrm{tot}}$

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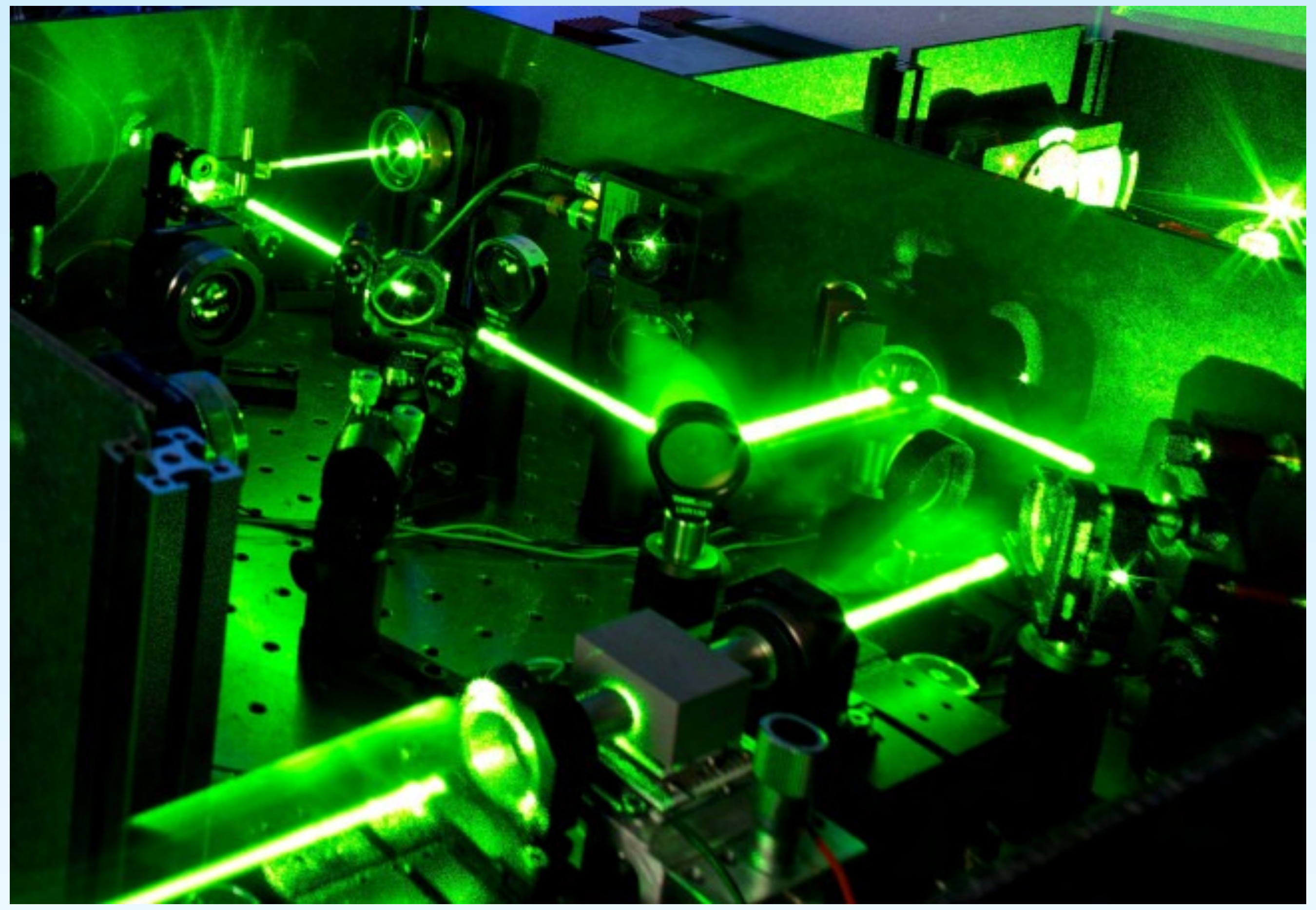

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**Assumption:**

The unknown state $\rho$ satisfies $\boxed{\mathrm{Tr}\left[\rho\hat{E}\right] \leq nE}$ for some *known* $E$

# Tomography of energy-constrained states

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**Theorem**

Let $\psi \in \mathcal{S}(n, E)$ be an **unknown** state. Then, a number $N = \tilde{\Theta}\left(\frac{E^n}{\varepsilon^{2n}}\right)$ of copies of $\psi$ are both **necessary** and **sufficient** to output $\tilde{\psi}$ such that

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CV tomography is ‘extremely inefficient’

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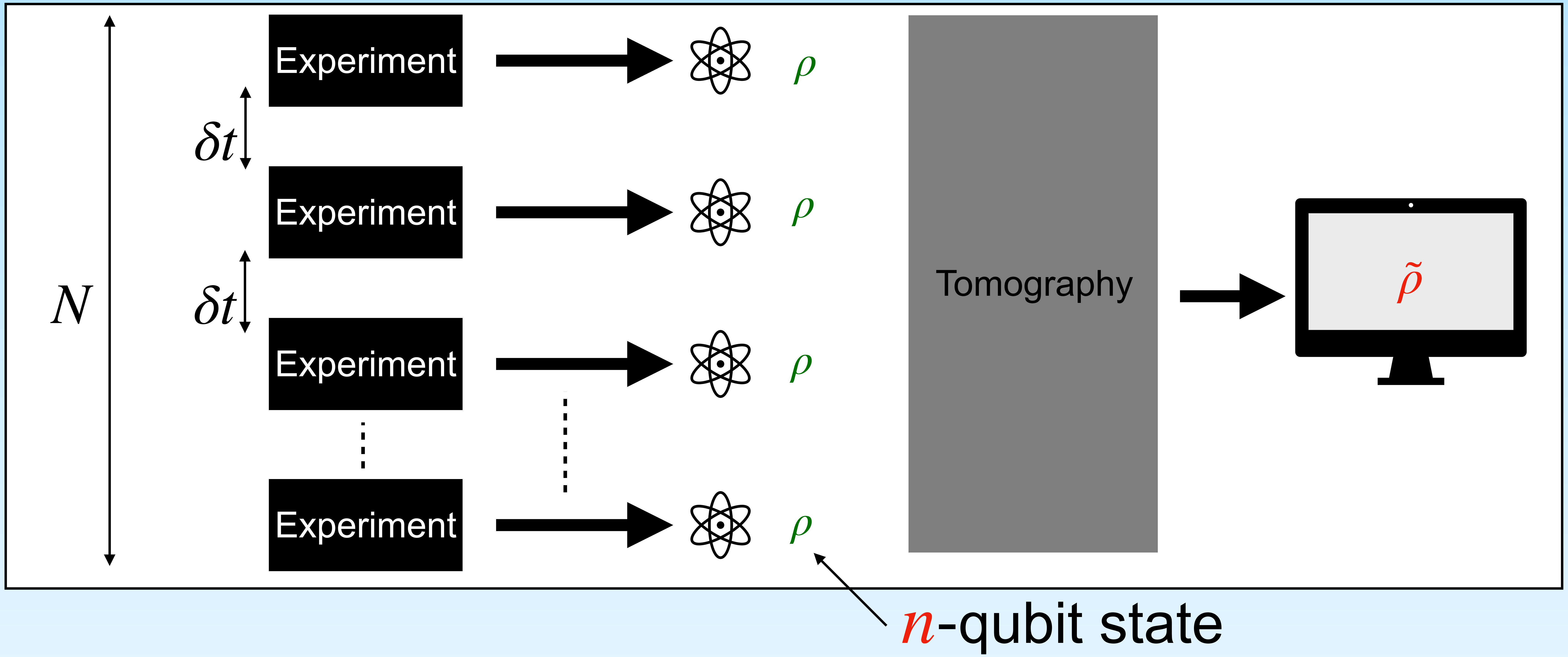


$\varepsilon = 0.1, \; \delta t = 1\,\mathrm{ns}$

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⟶ Total time $= 0.1\,\mathrm{ms}$

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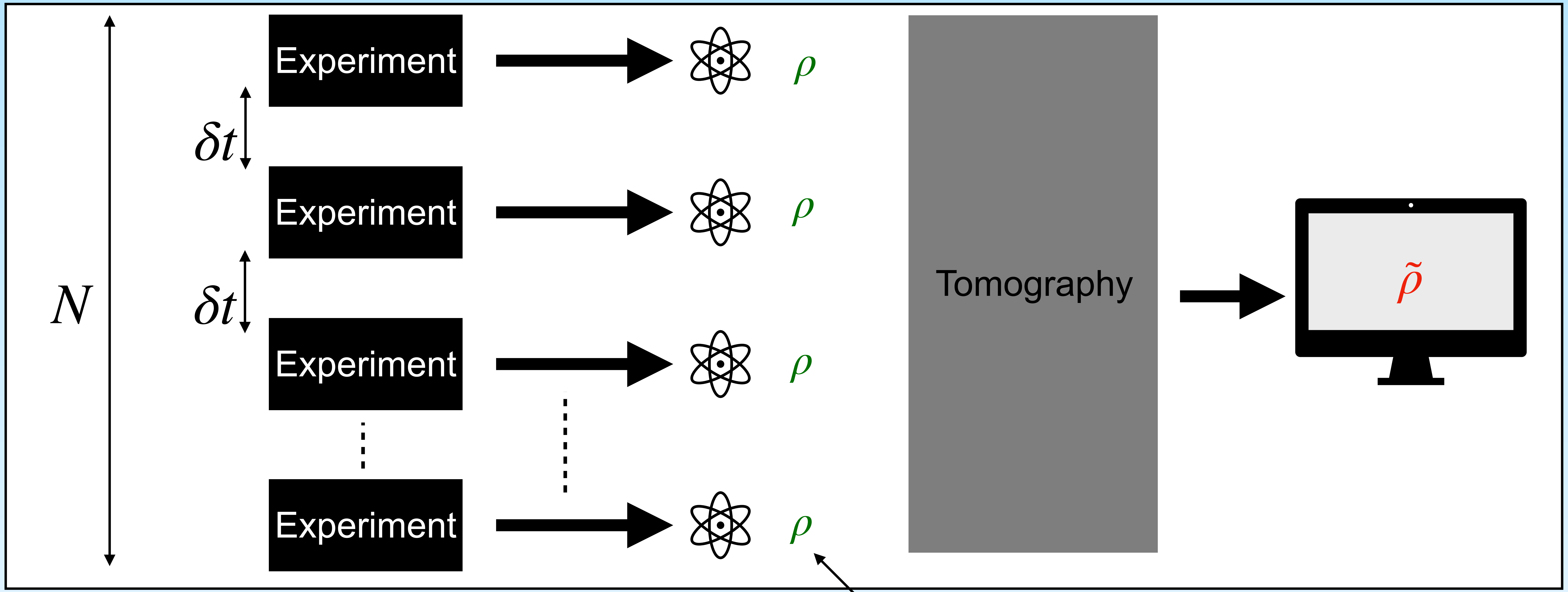


$n$-mode state with energy per mode $\leq E$

$\varepsilon = 0.1,\ \delta t = 1\,\text{ns}$

**$n$-qubit tomography: $n = 10$**

⟶ Total time $= 0.1\,\text{ms}$

**CV tomography: $n = 10,\ E = 1$**

⟶ Total time $= 3000\,\text{years}$

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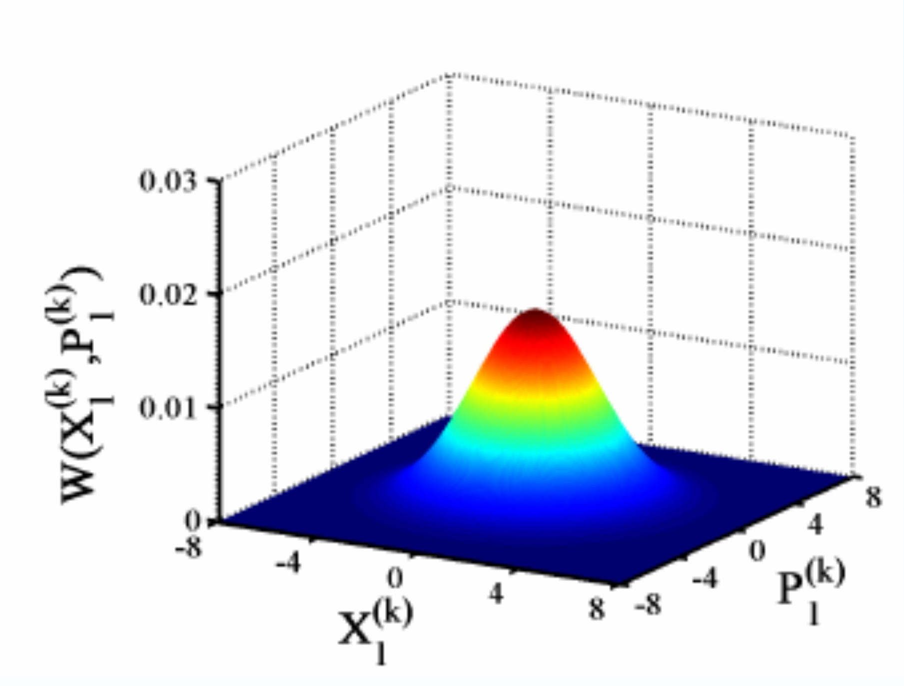
- $\mathrm{m}(\rho)$ **(first moment)**

- $V(\rho)$ **(covariance matrix)**

**Fact**

A Gaussian state $\rho$ is *uniquely* identified by $\mathrm{m}(\rho)$ and $V(\rho)$.




**Learning quantum states of continuous variable systems,** arXiv:2405.01431 (2024)

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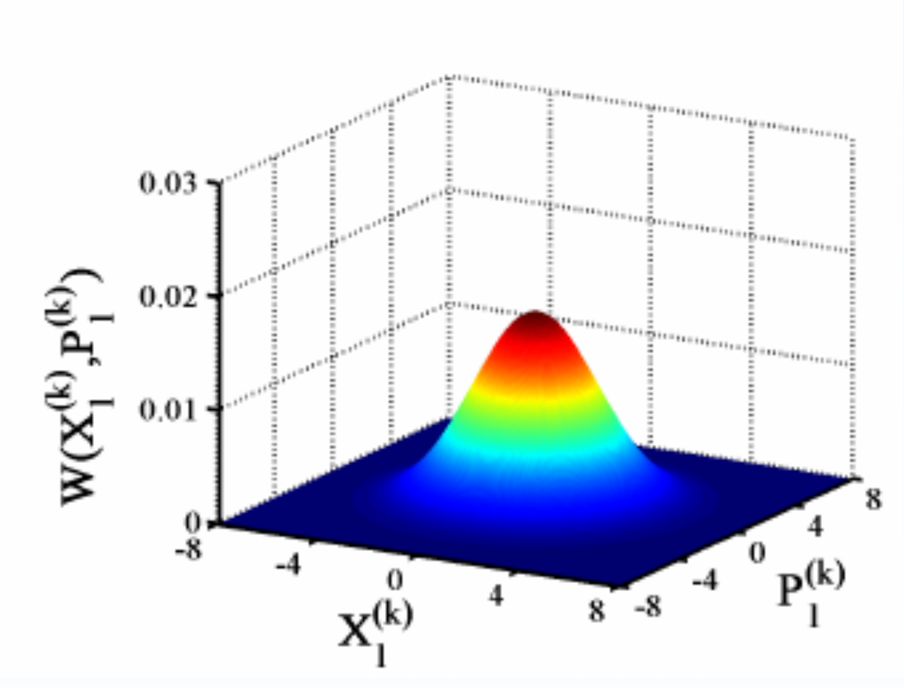

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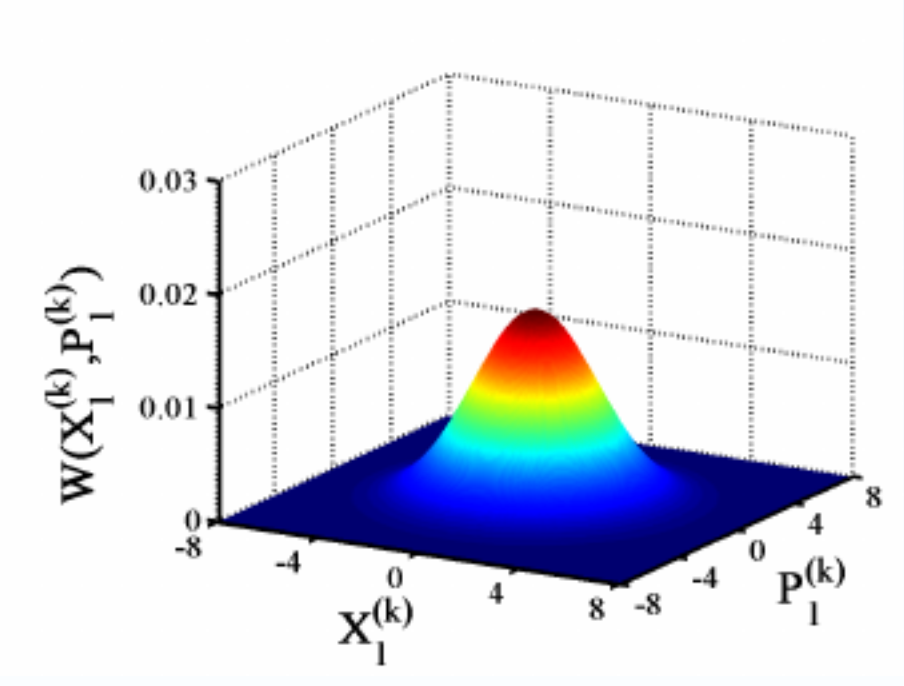


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If we estimate $m(\rho)$ and $V(\rho)$ of an unknown Gaussian state $\rho$ with precision $\varepsilon$, what is the resulting **trace distance error** on the state?

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$$\rho_{\mathrm{m},V} := \text{Gaussian state}$$

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**Theorem**

Let $\mathrm{Tr}[\rho_{\mathrm{m},V}\hat{E}] \leq E$ and $\mathrm{Tr}[\rho_{\mathrm{t},W}\hat{E}] \leq E$. Then:

$$\frac{1}{2}\|\rho_{\mathrm{m},V} - \rho_{\mathrm{t},W}\|_1 \leq \sqrt{2(E+1)}\left(\|\mathrm{m}-\mathrm{t}\|_2 + \sqrt{2\|V-W\|_1}\right)$$

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$$\frac{1}{2}\|\rho_{\mathrm{m},V} - \rho_{\mathrm{t},W}\|_1 \geq \frac{1}{200}\max\left(\min\left(1, \frac{\|\mathrm{m}-\mathrm{t}\|_2}{\sqrt{4E+1}}\right), \min\left(1, \frac{\|V-W\|_2}{4E+1}\right)\right)$$

$O(\varepsilon) \leq$ **Trace distance Error** $\leq O(\sqrt{\varepsilon})$

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In August, Holevo has found a new upper bound:

AS Holevo - arXiv preprint arXiv:2408.11400, 2024

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But still this!

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(*Tight* inequality) [Bittel L., Mele F.A., Mele A.A., Tirone S., Lami L., *Optimal estimates of trace distance between bosonic Gaussian states and applications to learning* (arXiv:2411.02368)]

$$\frac{1}{2}\|\rho_{\mathrm{m},V} - \rho_{\mathrm{t},W}\|_1 \leq \frac{1+\sqrt{3}}{8}\mathrm{Tr}\left[\,|V-W|\,\Omega^{\intercal}\left(\frac{V+W}{2}\right)\Omega\right] + \sqrt{\frac{\min(\|V\|_\infty, \|W\|_\infty)}{2}}\|\mathrm{m}-\mathrm{t}\|_2$$

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*Hölder's inequality* like this: $\le \frac{\|V\|_\infty + \|W\|_\infty}{2}\|V-W\|_1$

But also like this: $\le \frac{\mathrm{Tr}V + \mathrm{Tr}W}{2}\|V-W\|_\infty$

**Trace distance Error** $= O(\varepsilon)$

**Theorem** Tomography of **Gaussian states** is efficient

Let $\rho$ be an **unknown** $n$ mode Gaussian state with $\mathrm{Tr}[\rho\hat{E}] \leq nE$. Then, a number

$$N = O\left(\frac{n^7 E^4}{\varepsilon^4}\log\left(\frac{n^2}{\delta}\right)\right) = \mathrm{poly}(n)$$

of state copies suffices to output $\tilde{\rho}$ such that $\Pr\left[d_{\mathrm{tr}}(\tilde{\rho}, \rho) \leq \varepsilon\right] \geq 1 - \delta$

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Improvements in our new paper!

[Bittel L., Mele F.A., Mele A.A., Tirone S., Lami L., *Optimal estimates of trace distance between bosonic Gaussian states and applications to learning* (arXiv:2411.02368)]

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2. CV systems ✓
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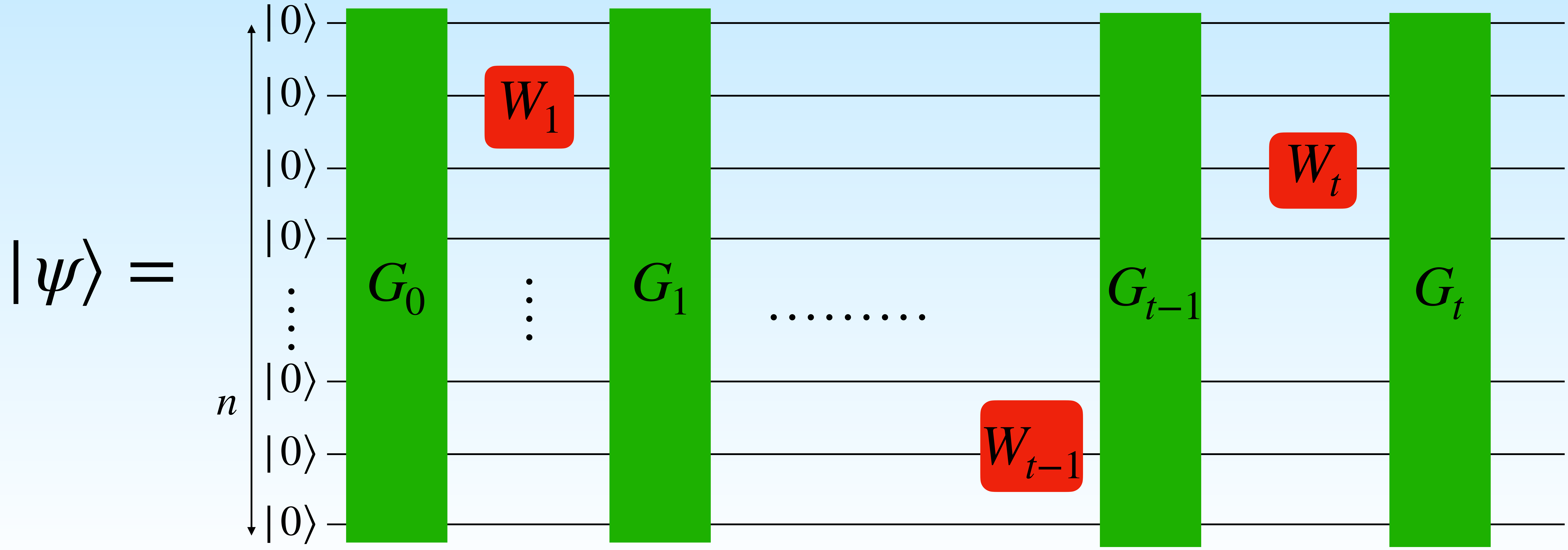
## Definition

An $n$-mode state $|\psi\rangle$ is a "$t$**-doped Gaussian state**" if it is of the form

$$|\psi\rangle = G_t W_t \cdots G_1 W_1 G_0 |0\rangle^{\otimes n}$$

Gaussian unitaries ($G_t, G_1, G_0$)

single-mode (non-Gaussian) unitaries ($W_t, W_1$)



**Theorem** (compression of non-Gaussianity)

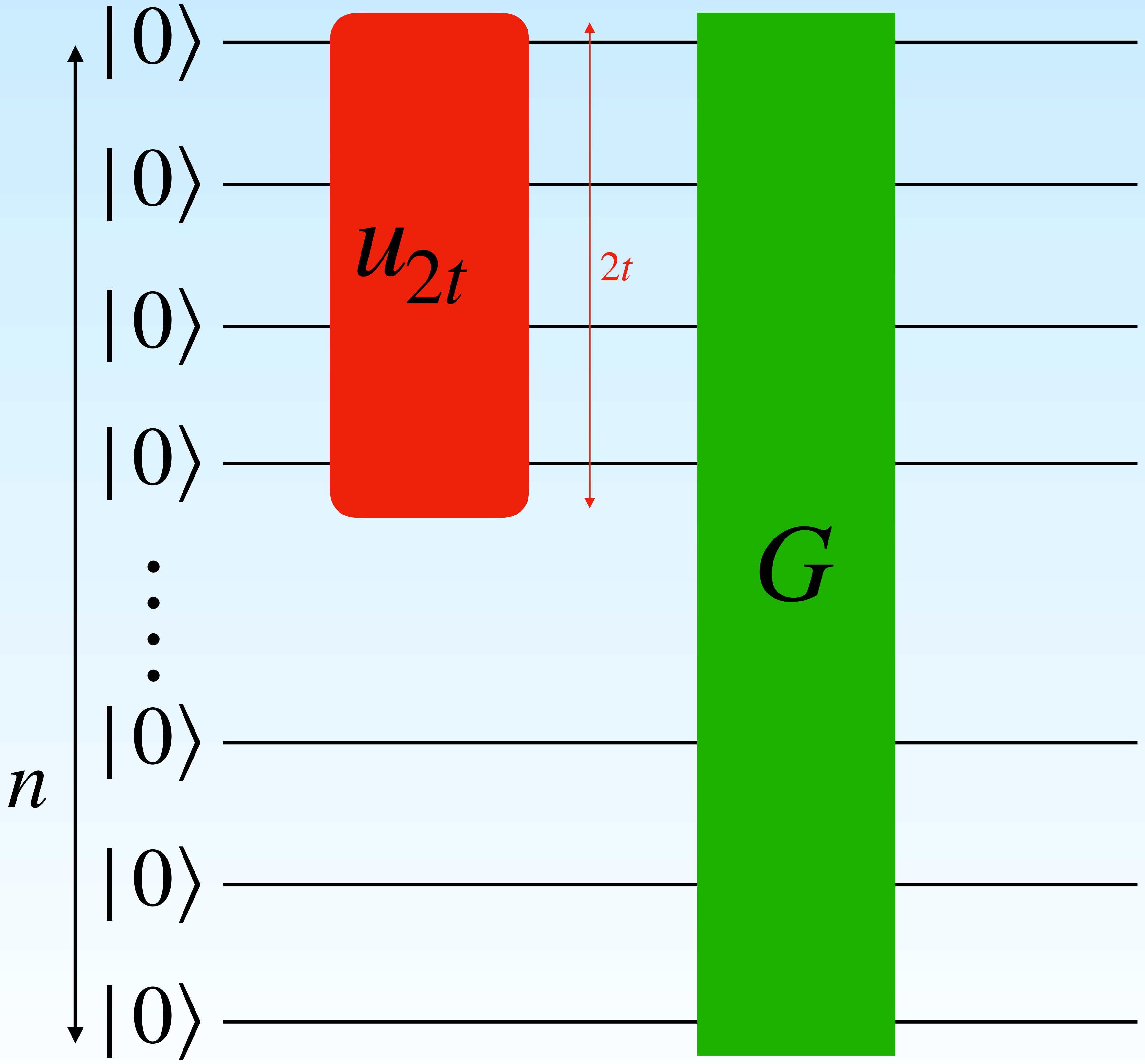
Any $t$-**doped Gaussian state** $|\psi\rangle$ can be written as

$$|\psi\rangle = G(u_{2t} \otimes \mathbf{1}_{n-2t})\,|0\rangle^{\otimes n}$$

Gaussian unitary

$2t$-mode (non-Gaussian) unitary

$$|\psi\rangle =$$

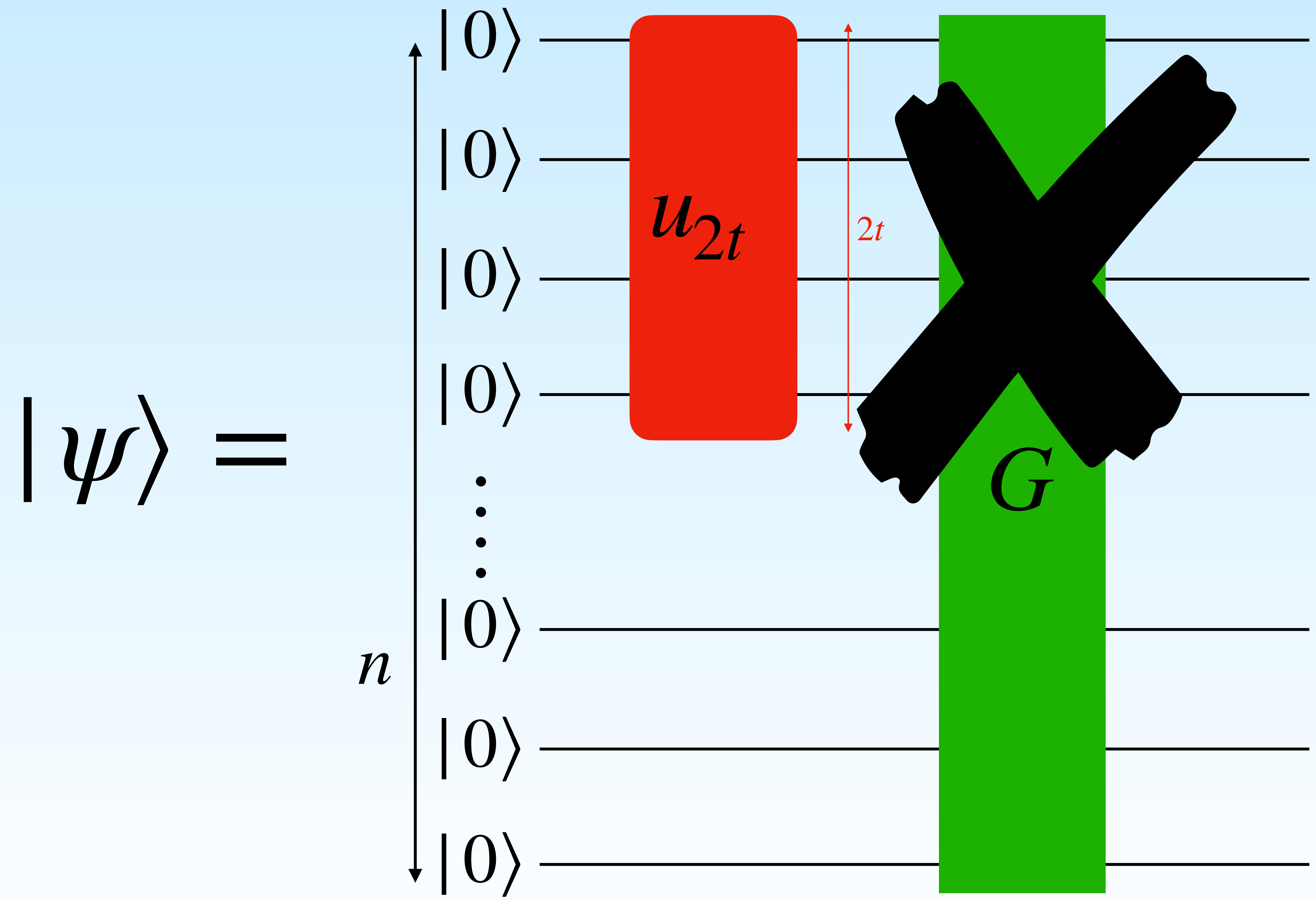



Analogous results available in different settings:

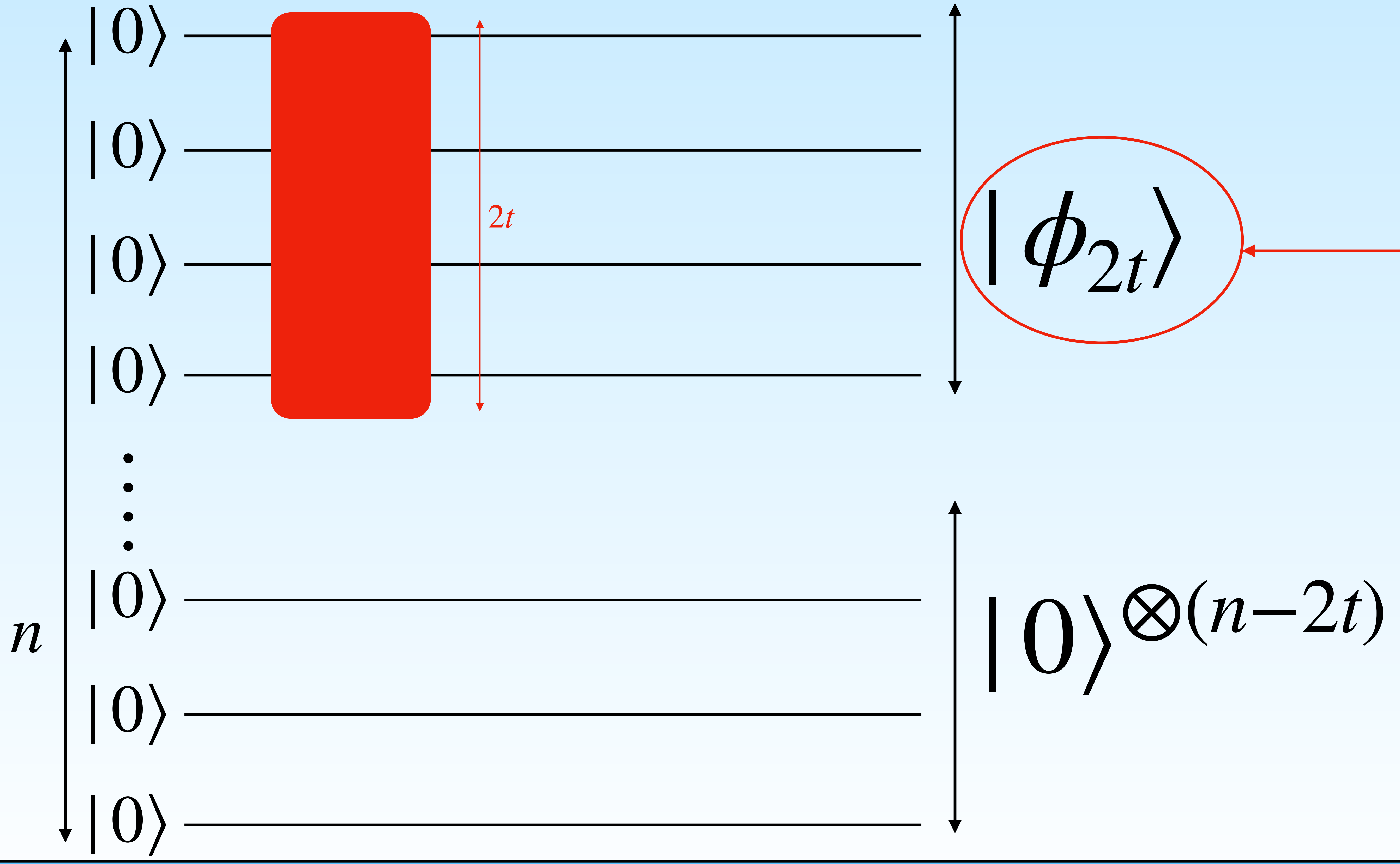
- *Fermionic* setting [Mele A. A., Herasymenko Y., *Efficient learning of quantum states prepared with few fermionic non-Gaussian gates* (2024)]
- *Clifford* setting [Leone L., Oliviero S.F.E., Hamma A., *Learning t-doped stabilizer states* (2023)] [Grewal S., Iyer V., Kretschmer W., Liang D., *Efficient Learning of Quantum States Prepared With Few Non-Clifford Gates* (2023)]

# **Idea of the tomography algorithm** (part 1)

By estimating $\mathrm{m}(|\psi\rangle\langle\psi|)$ and $V(|\psi\rangle\langle\psi|)$, one can learn and undo $G$



# Idea of the tomography algorithm (part 2)

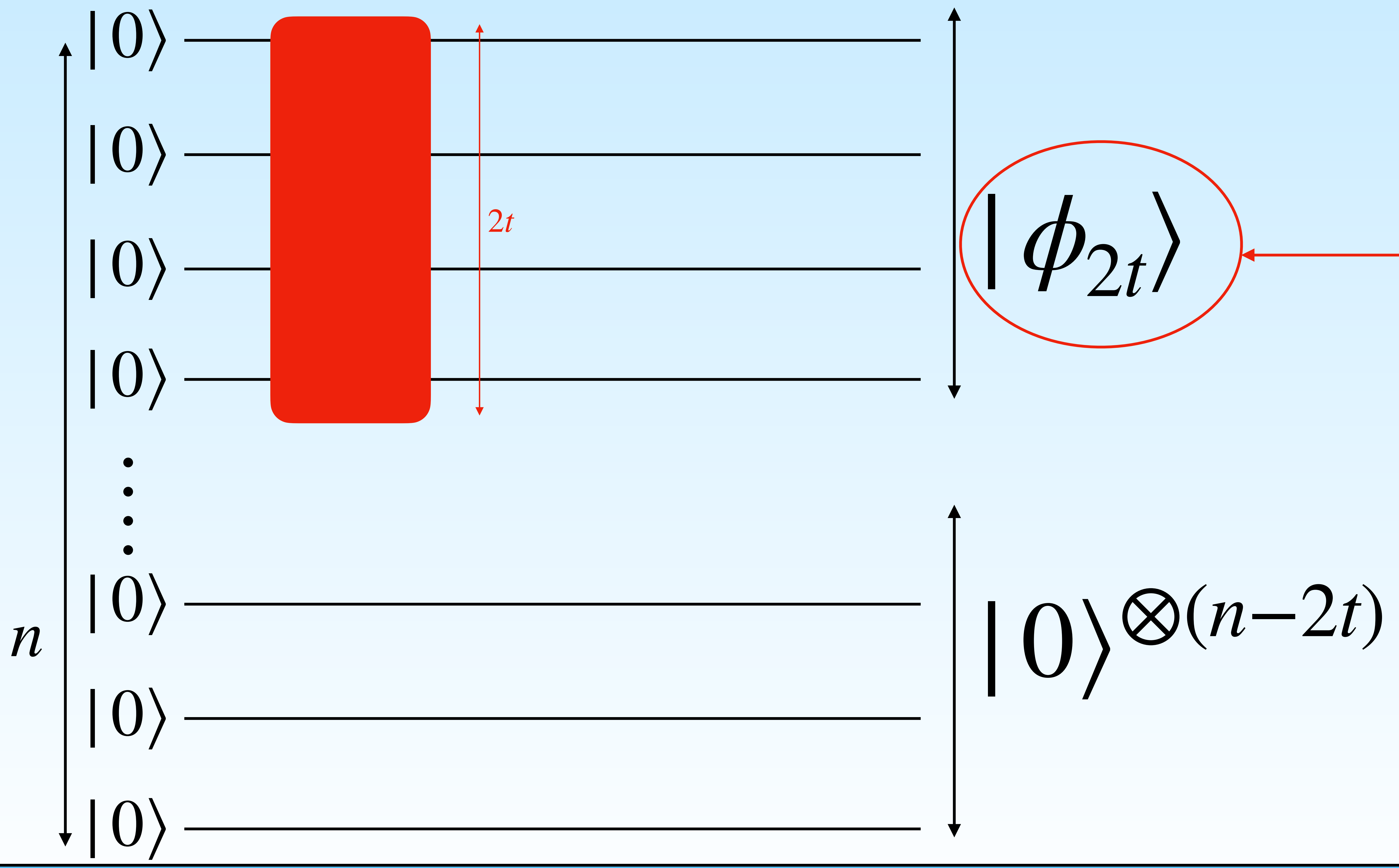



Perform full-state tomography on the first $2t$ modes

**Theorem**

If $t = O(1)$, tomography of $t$-**doped Gaussian states** is efficient.

$N = O(n^t)$




Perform full-state tomography on the first $2t$ modes

# Outline

1. Quantum state tomography ✓
2. CV systems ✓
3. Quantum state tomography of CV systems:
   A. Energy-constrained states ✓
   B. Gaussian states ✓
   C. $t$-doped Gaussian states ✓

# Conclusions

First investigation of CV tomography with rigorous guarantees wrt trace distance.

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# Conclusions

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- Trade-off between "efficiency in tomography" and "non-Gaussianity":
  ($t$-doped Gaussian states)
- Technical tools of independent interest:
  - Bounds on the trace distance between Gaussian states;
  - *Effective dimension* and *effective rank* of energy-constrained states;
  - Decomposition of $t$-doped Gaussian unitaries/states.

# Open problems

- Optimal sample-complexity for energy-constrained mixed states;
- Optimal sample-complexity for Gaussian states;
- Property testing of Gaussian states
- Bosonic channels:
    A. Tomography of arbitrary bosonic channels;
    B. Tomography of bosonic Gaussian channels.
- Classical simulability of t-doped Gaussian states

# Thank you!

# Backup slides

$\hat{\mathrm{R}} := \left(\hat{x}_1, \hat{p}_1, \cdots, \hat{x}_n, \hat{p}_n\right)^{\mathrm{t}}$ (**quadrature** operator vector)

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**Definition** (Informal)

$\rho$ is a "**Gaussian state**" if

$$\rho = \frac{e^{-\beta \hat{H}}}{\mathrm{Tr}[e^{-\beta \hat{H}}]}$$

for some $\beta \in (0,\infty]$ and some (quadratic) Hamiltonian $\hat{H}$ of the form

$$\hat{H} := (\hat{\mathrm{R}} - \mathrm{m})^{\mathrm{t}}\, h\, (\hat{\mathrm{R}} - \mathrm{m}),$$

where $h \in \mathbb{R}^{2n,2n}$ is symmetric positive definite and $m \in \mathbb{R}^{2n}$.

- $\mathrm{m}(\rho) := \mathrm{Tr}[\rho \hat{\mathrm{R}}]$ **(first moment)**

- $V(\rho)$ **(covariance matrix)**

$$V_{ij}(\rho) := \mathrm{Tr}\left[\rho \left\{ \hat{R}_i - m_i(\rho)\mathbf{1}, \hat{R}_j - m_j(\rho)\mathbf{1} \right\}\right] \quad \forall i, j \in [2n]$$

$$\{A, B\} = AB + BA$$

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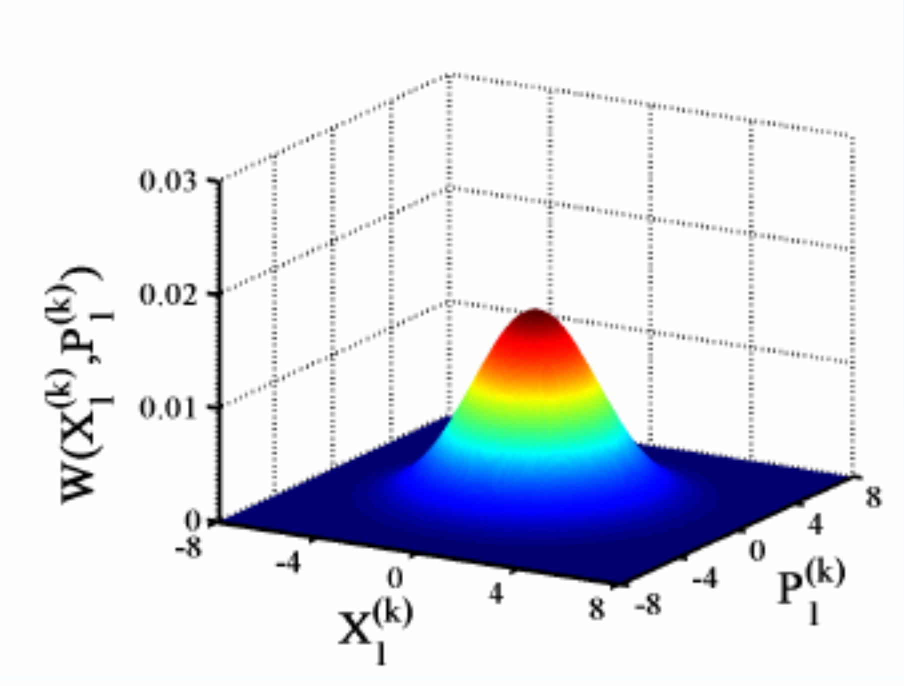
Estimated in lab via *Homodyne*

**Fact**

A Gaussian state $\rho$ is *uniquely* identified by $\mathrm{m}(\rho)$ and $V(\rho)$.




**Learning quantum states of continuous variable systems,** arXiv:2405.01431 (2024)

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Estimation of $\rho$ ← Estimated in lab via *Homodyne*

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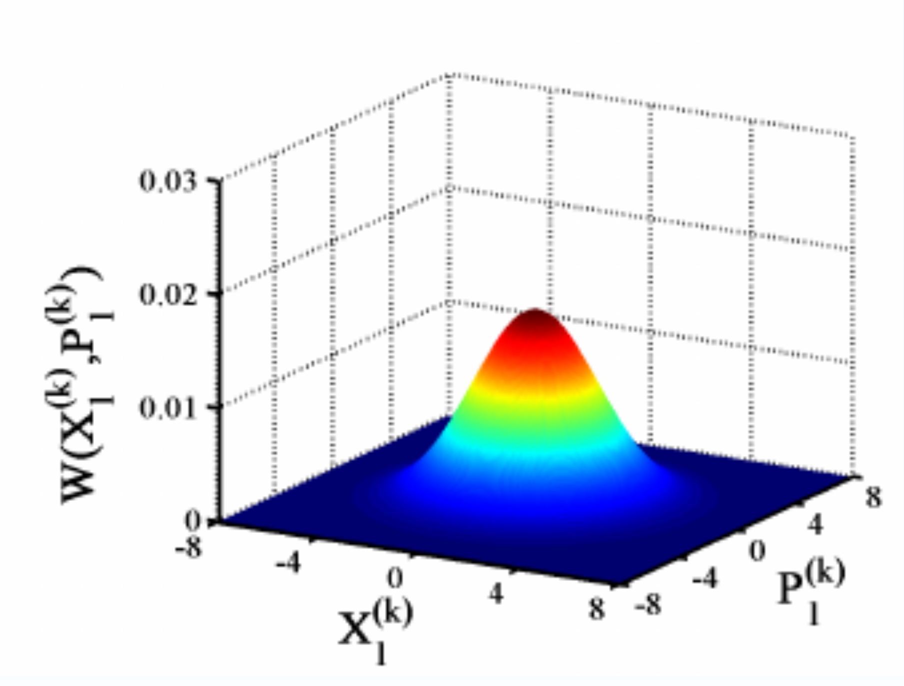

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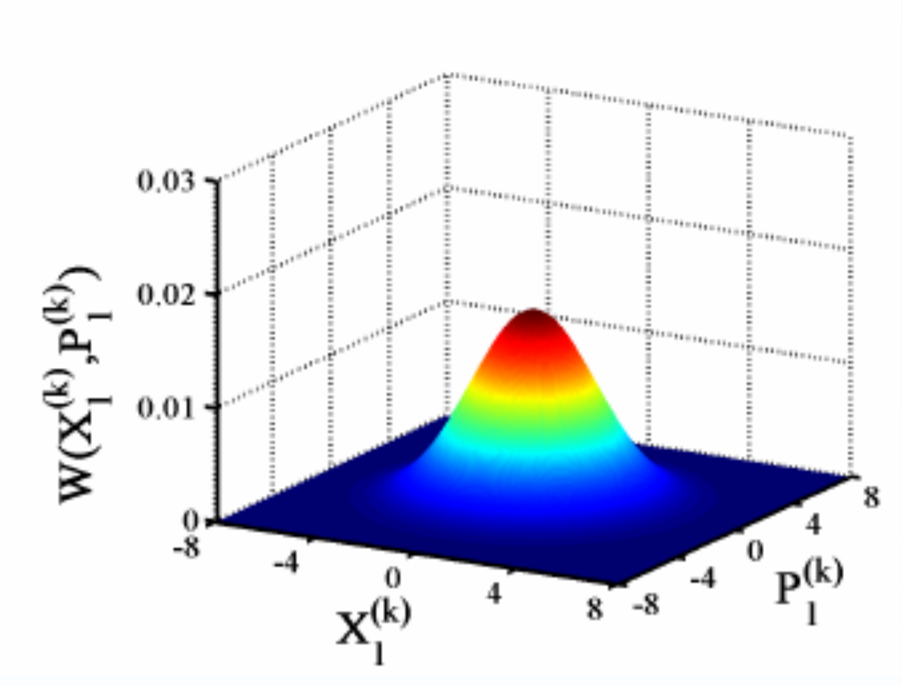
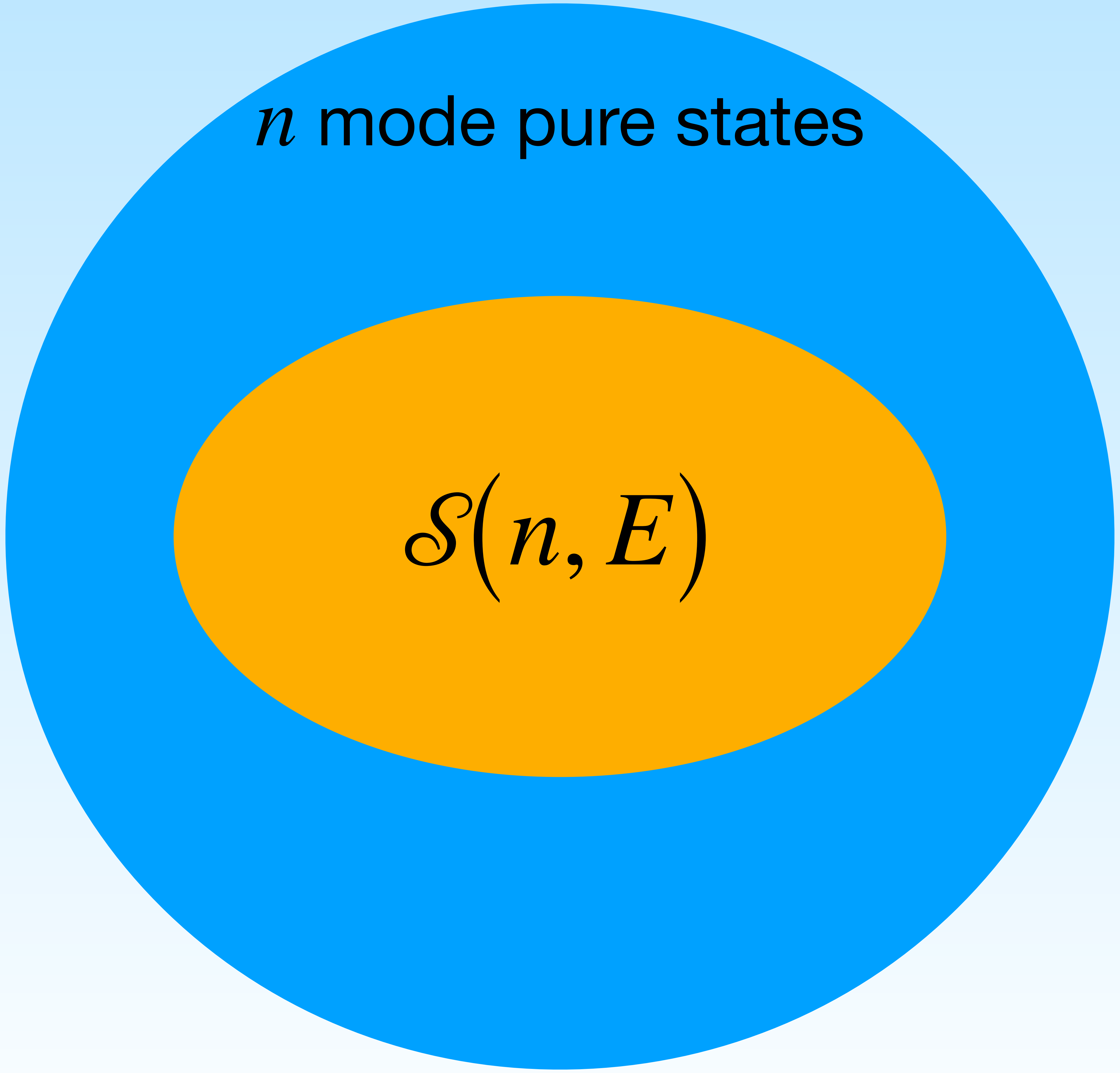


## **Proof sketch of** "Any tomography algorithm must satisfy $N \geq \tilde{\Theta}\left(\frac{E^n}{\varepsilon^{2n}}\right)$"




$$\mathcal{S}(n, E) := \left\{ n \text{ mode pure state } \psi : \operatorname{Tr}\left[\psi \hat{E}\right] \leq nE \right\}$$

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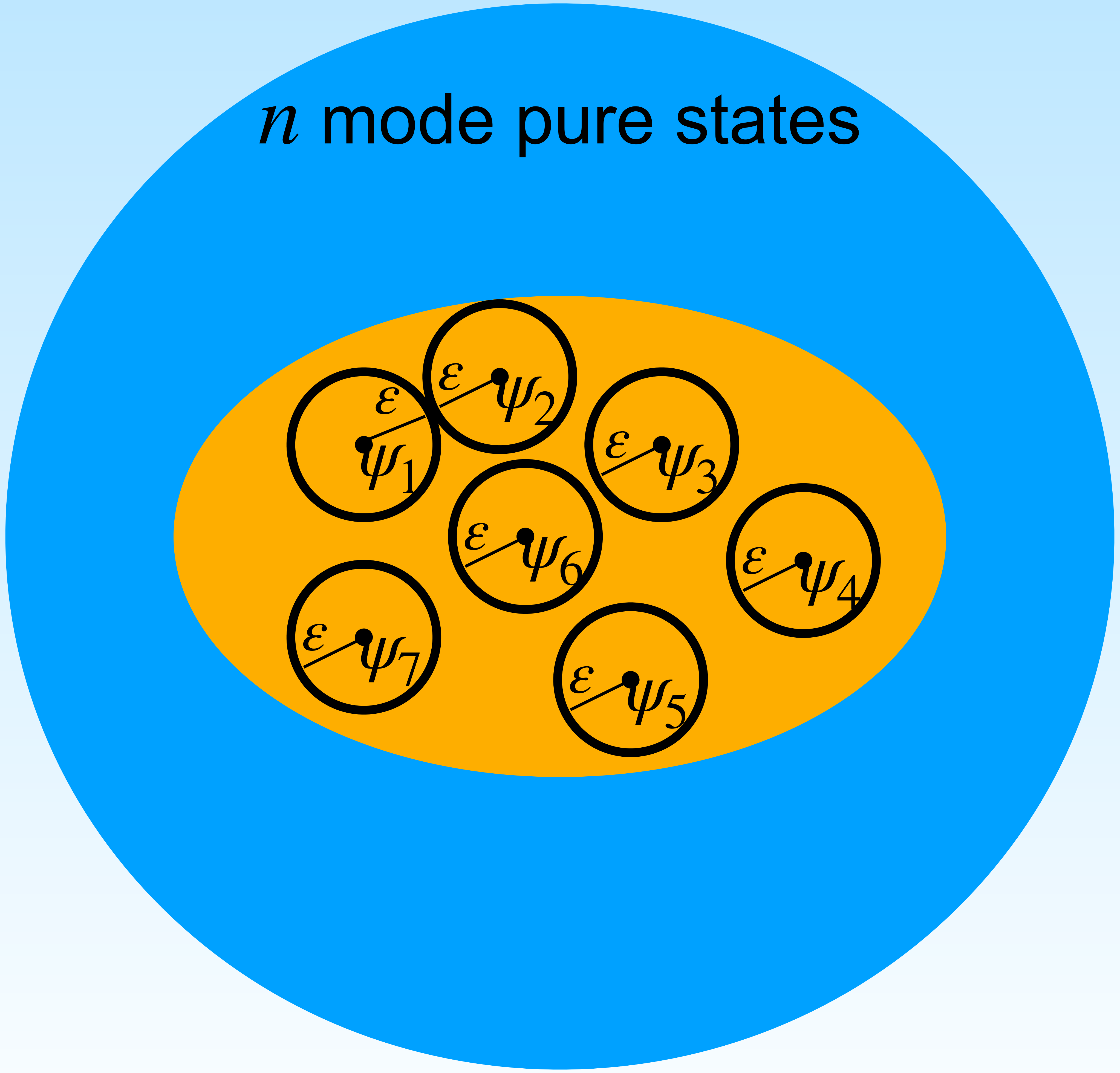



**$\varepsilon$-packing net:**

We construct $M$ *energy-constraint* states

$$\{\psi_1, \psi_2, \cdots, \psi_M\} \subseteq \mathcal{S}(n, E)$$

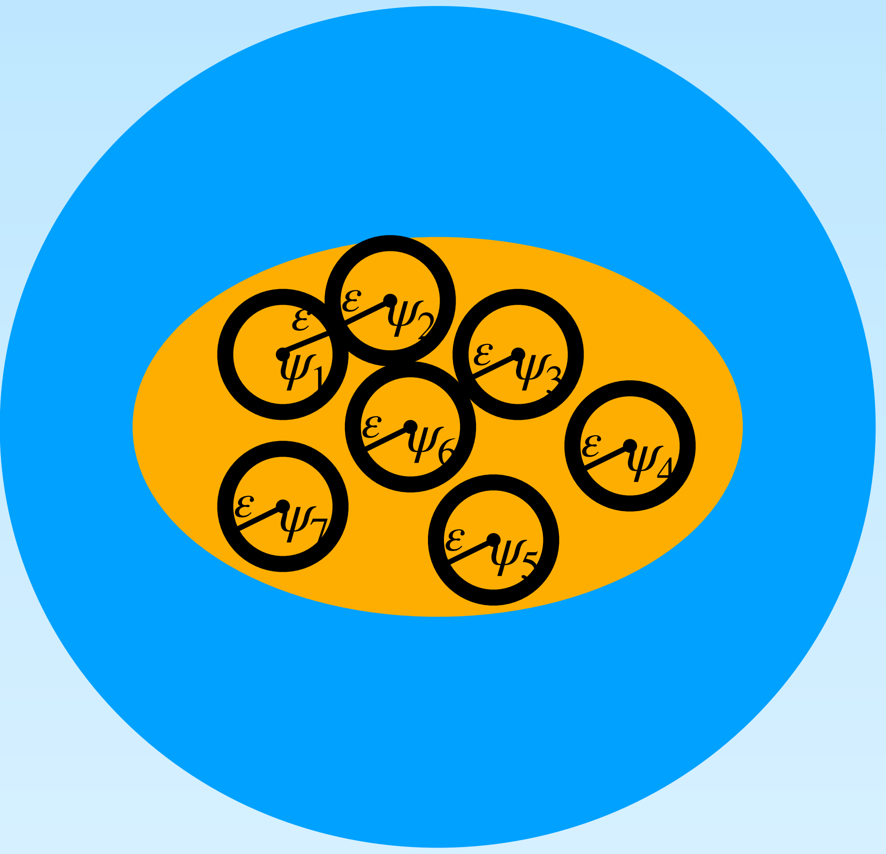
such that

$$d_{\mathrm{tr}}(\psi_i, \psi_j) > 2\varepsilon \quad \forall i \neq j \in [M]$$

## **Proof sketch of** "Any tomography algorithm must satisfy $N \geq \tilde{\Theta}\left(\frac{E^n}{\varepsilon^{2n}}\right)$"

Alice

Bob



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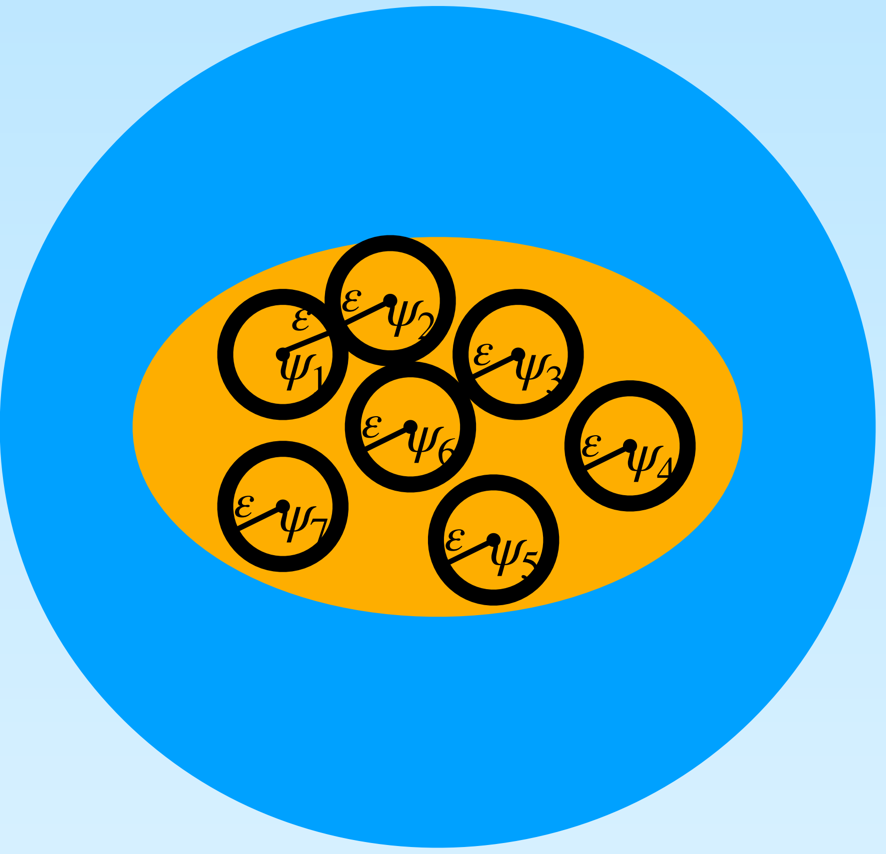
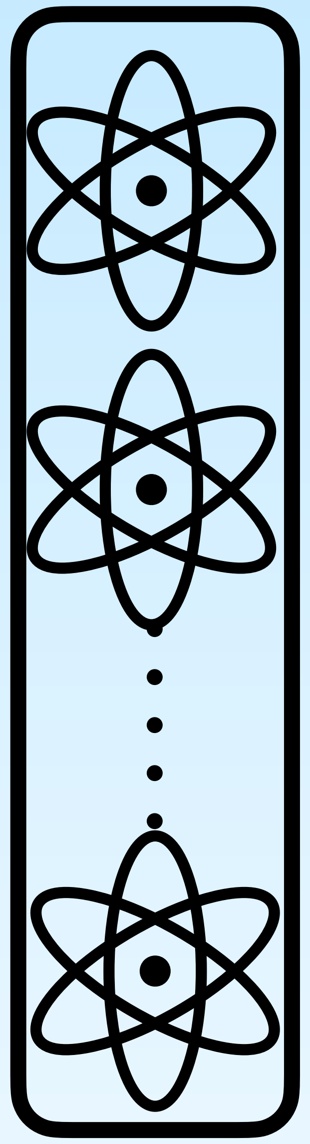
Bob



$\psi_i^{\otimes N}$

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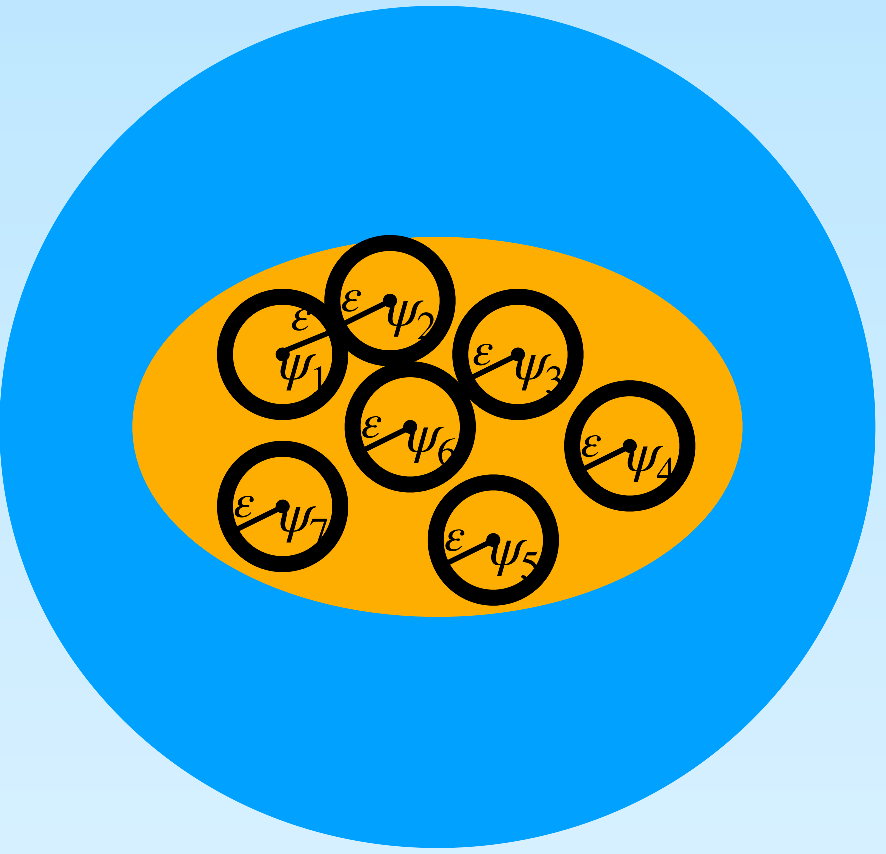
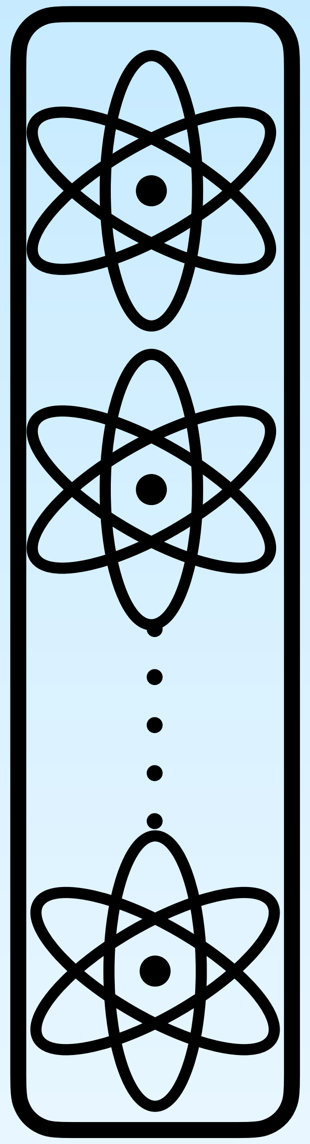


Bob



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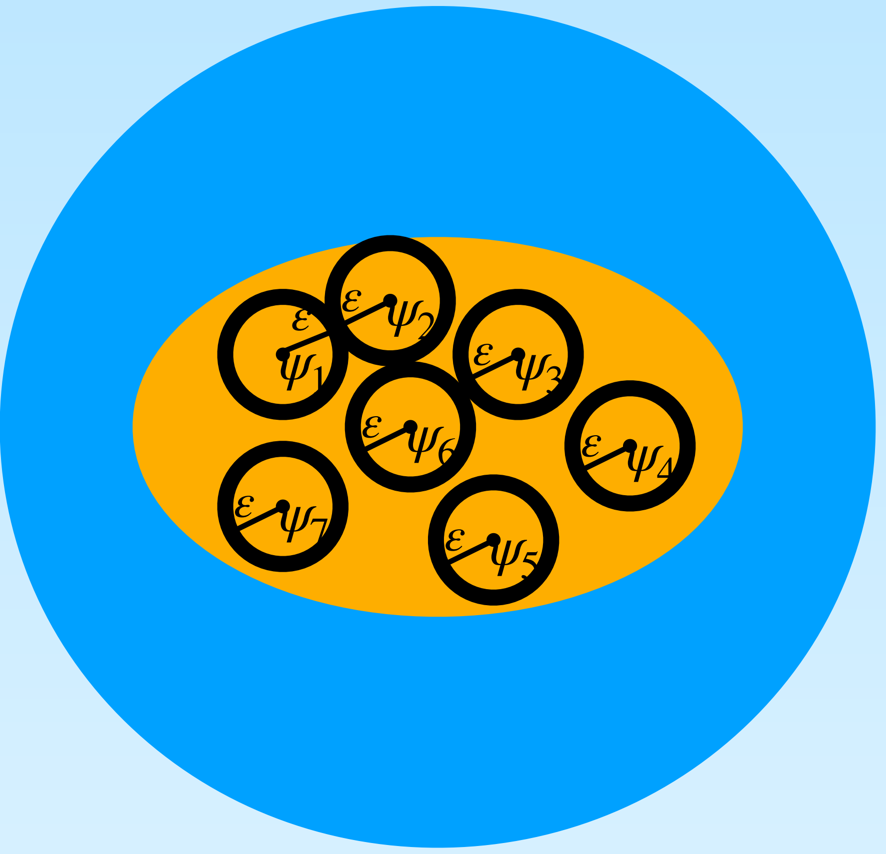

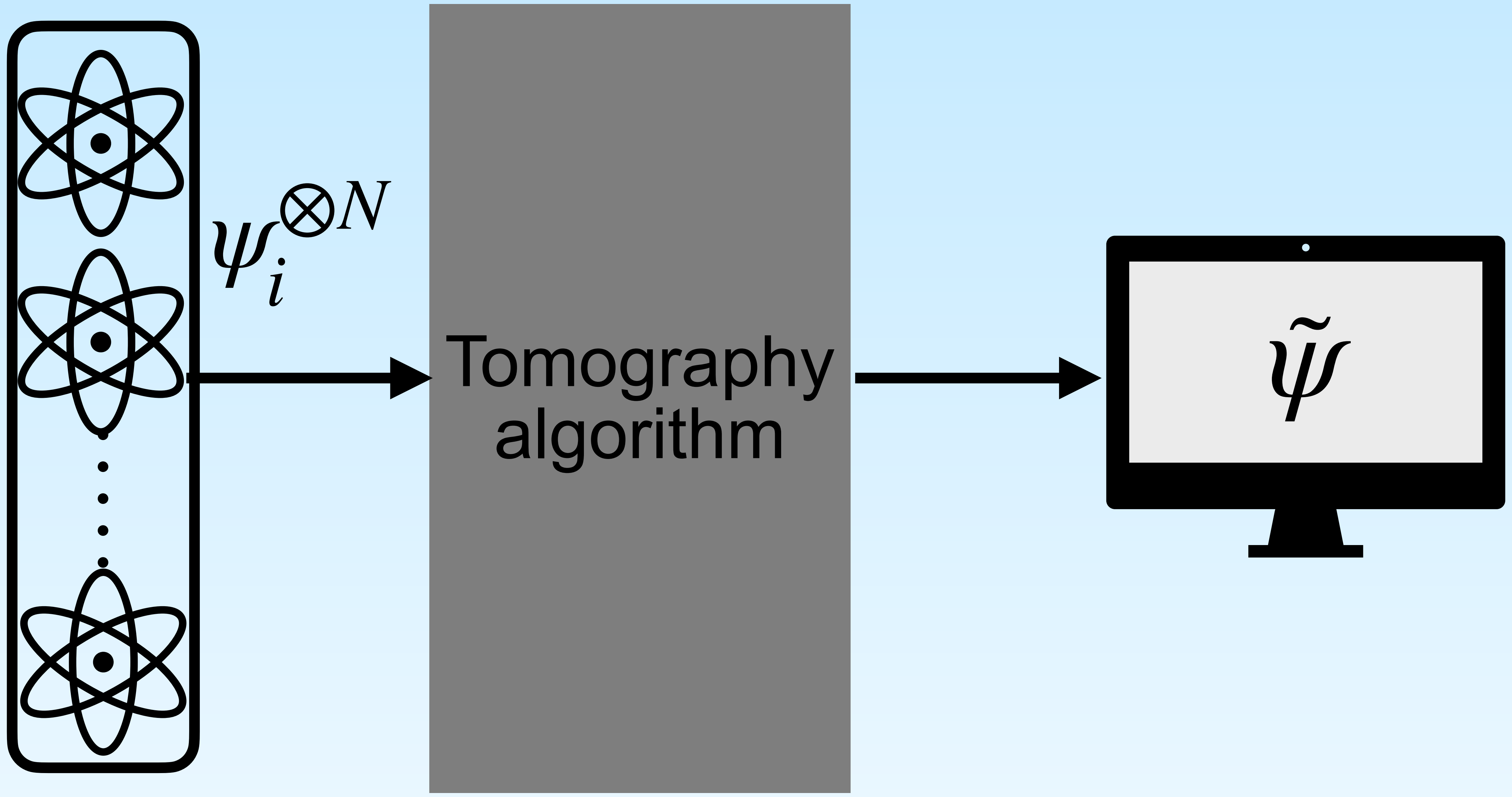


**Learning quantum states of continuous variable systems,** arXiv:2405.01431 (2024)

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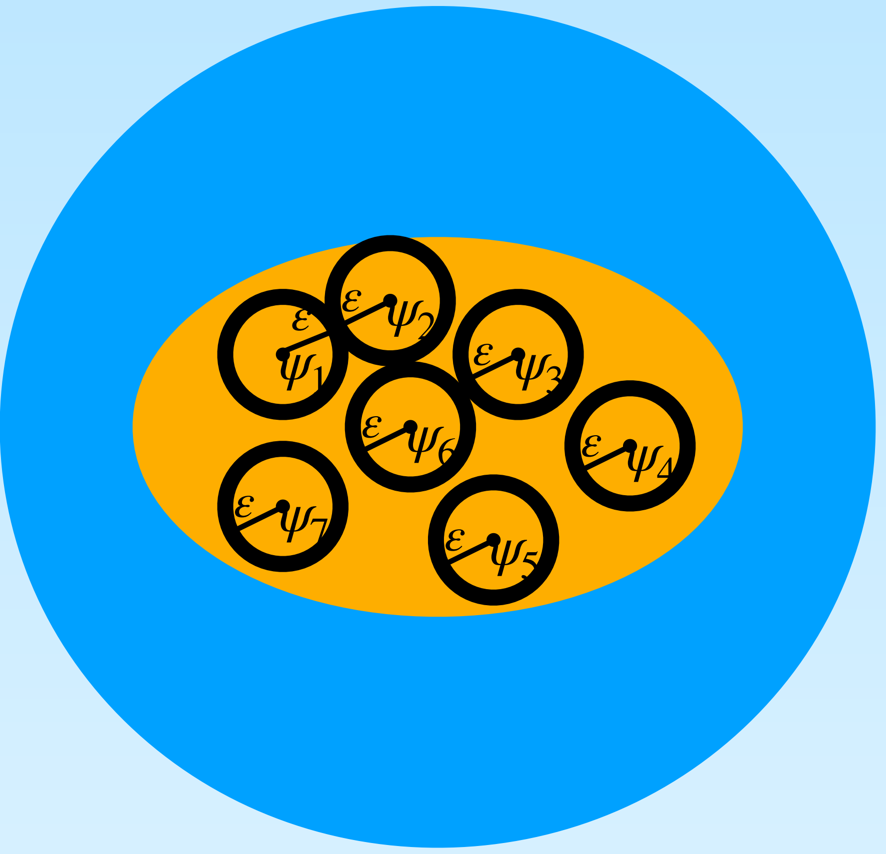

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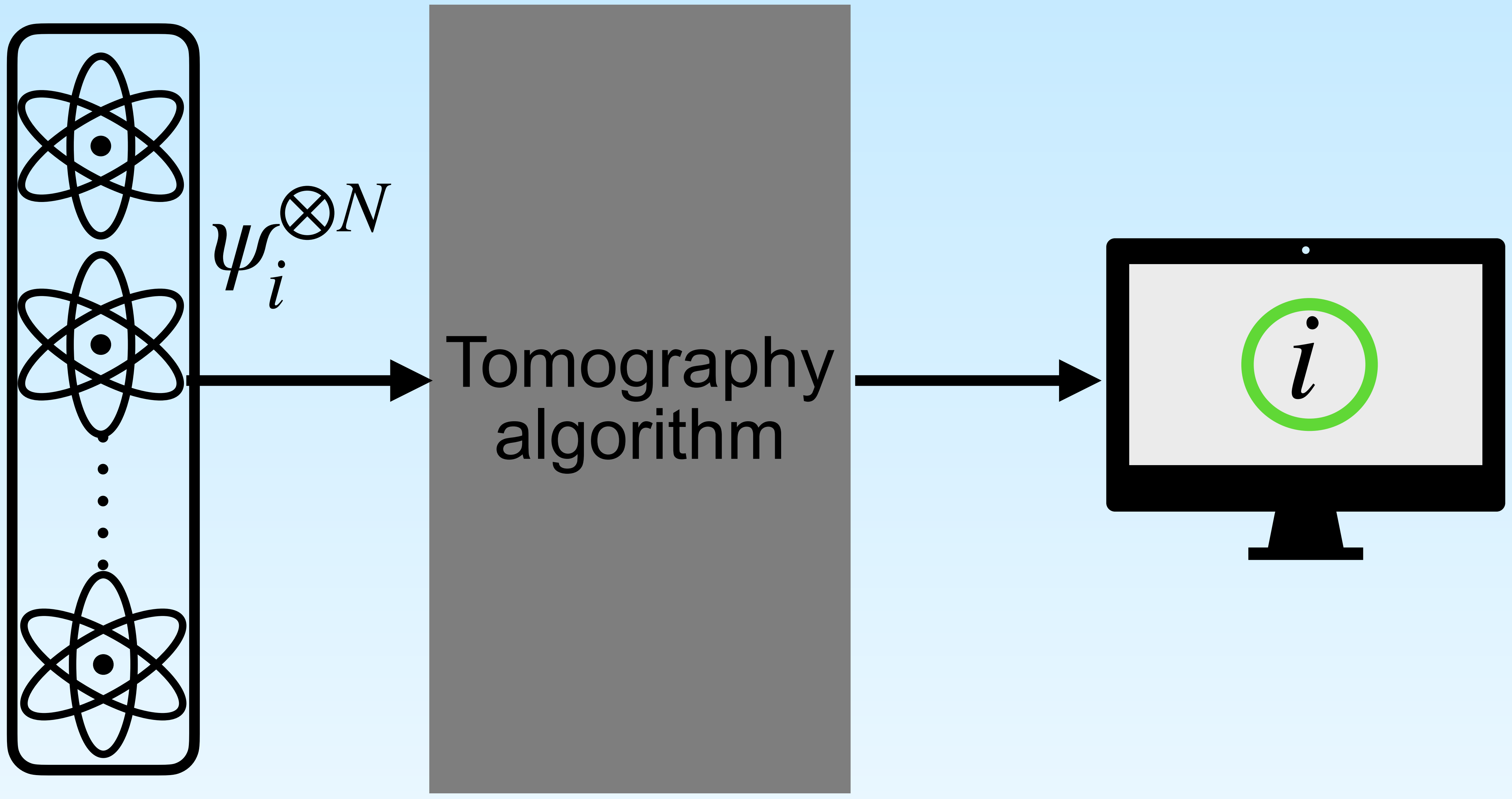
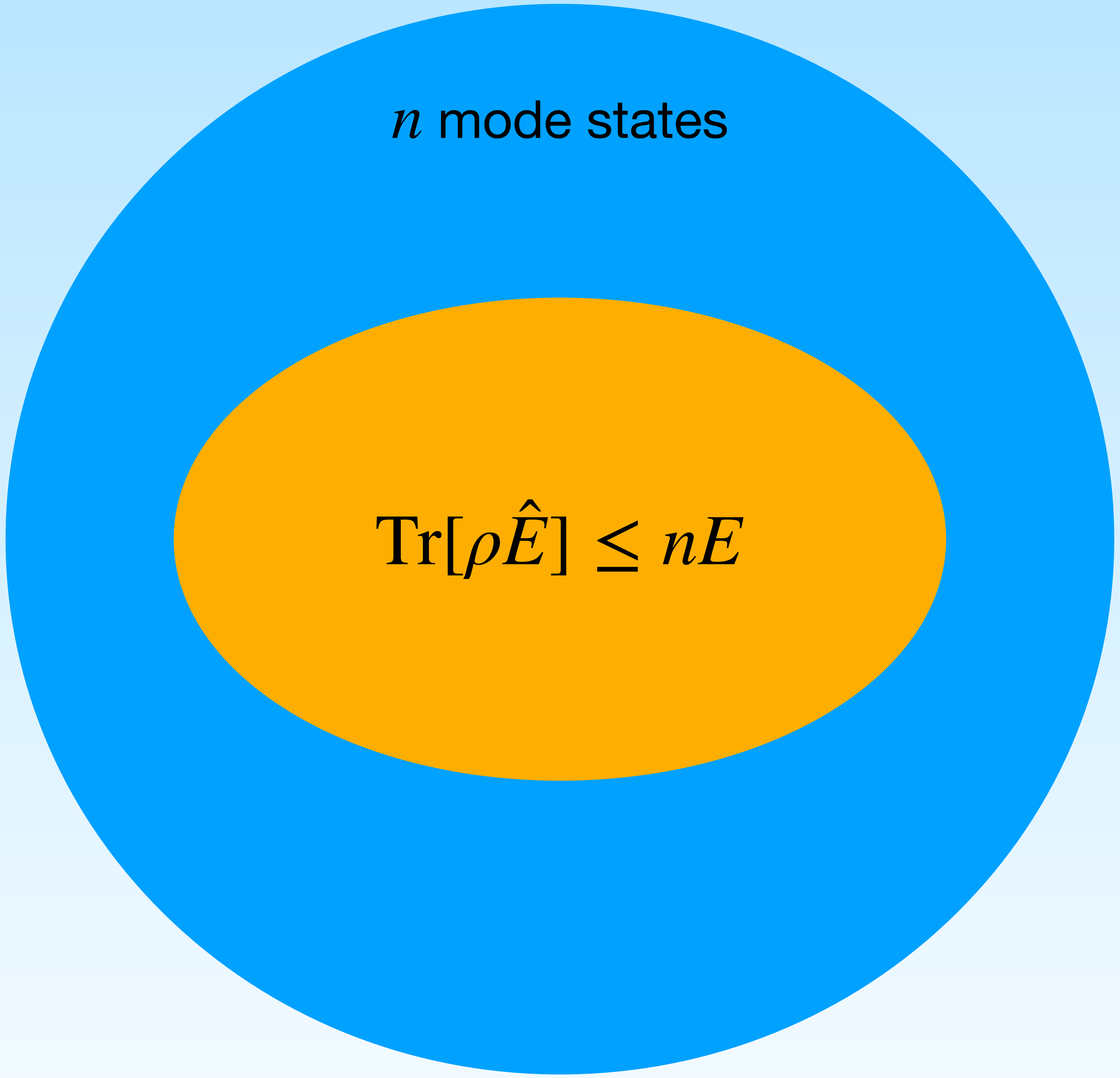


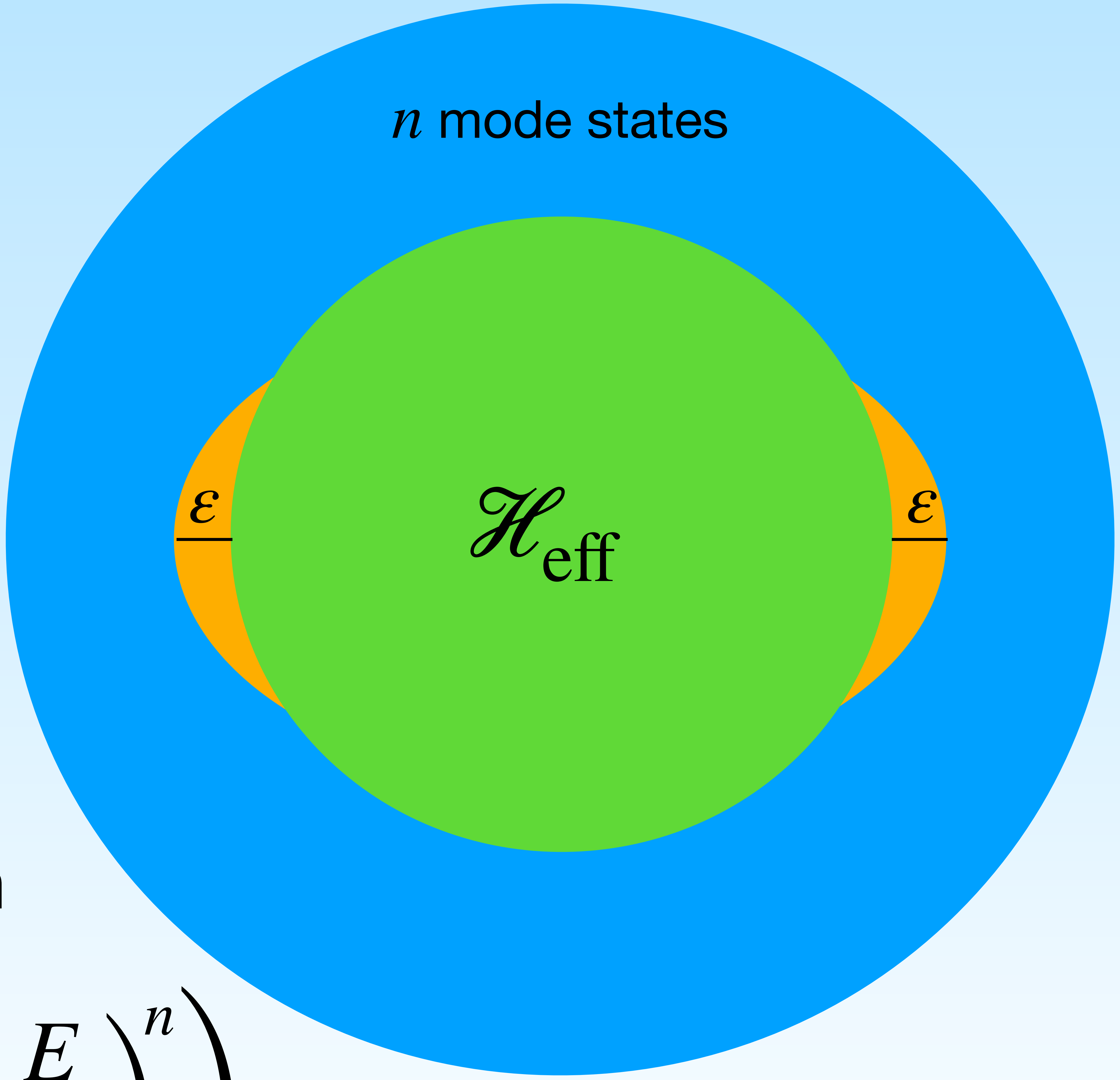

Holevo bound $\longrightarrow$ $N \geq \tilde{\Theta}(\log_2 M) = \tilde{\Theta}\left(\left(\frac{E}{\varepsilon^2}\right)^n\right)$

## Proof sketch of "There is a tomography algorithm with $N = \tilde{\Theta}\left(\frac{E^n}{\varepsilon^{2n}}\right)$"



**Proof sketch of** "There is a tomography algorithm with $N = \tilde{\Theta}\left(\frac{E^n}{\varepsilon^{2n}}\right)$"

$$\dim \mathscr{H}_{\text{eff}} = \tilde{\Theta}\left(\left(\frac{E}{\varepsilon^2}\right)^n\right)$$




Hence, quantum state tomography is achievable with

$$N = \tilde{\Theta}\left(\frac{\dim \mathscr{H}_{\text{eff}}}{\varepsilon^2}\right) = \tilde{\Theta}\left(\left(\frac{E}{\varepsilon^2}\right)^n\right).$$

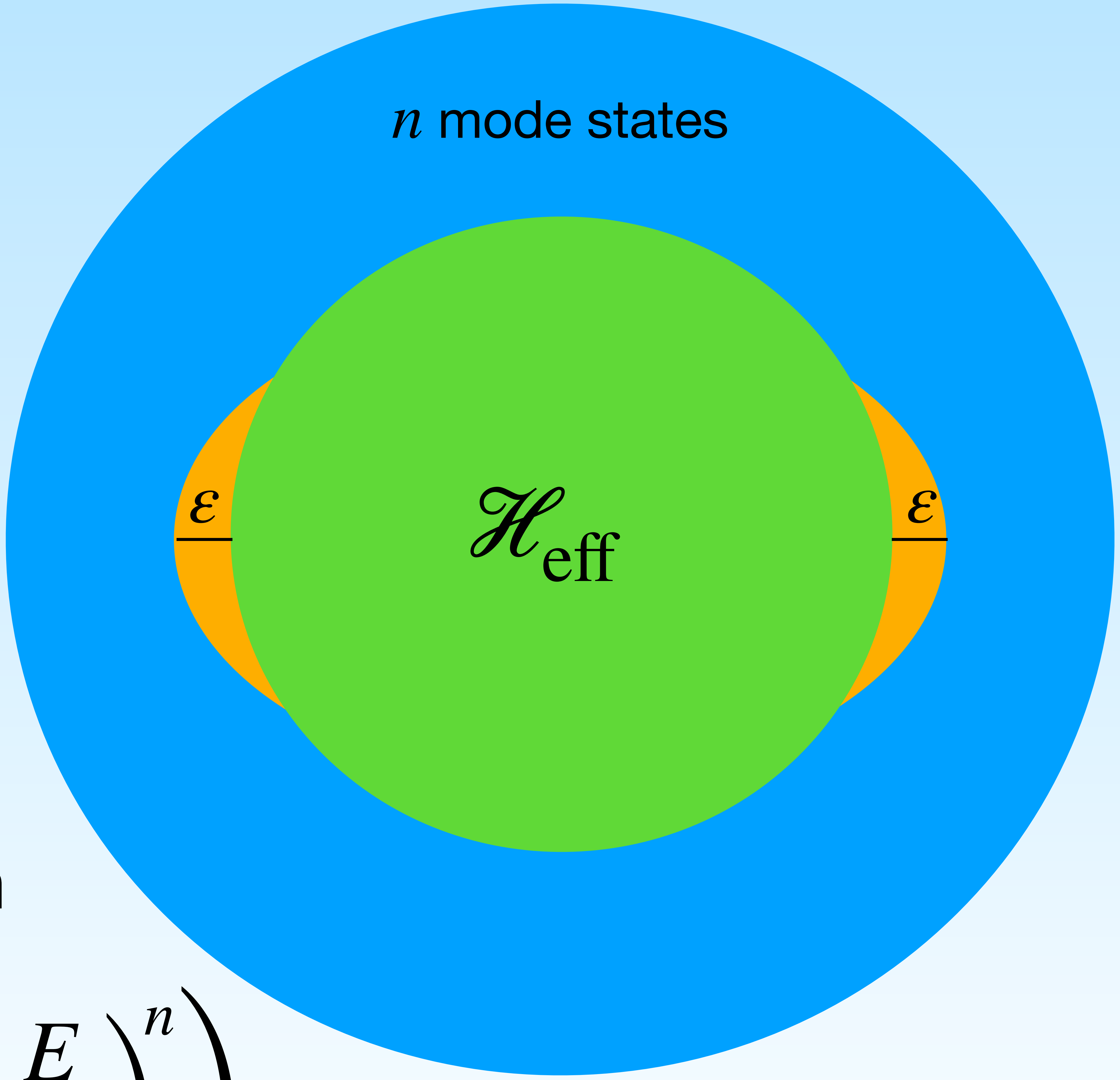
## **Proof sketch of** "There is a tomography algorithm with $N = \tilde{\Theta}\left(\frac{E^n}{\varepsilon^{2n}}\right)$"

**Lemma**

$$\mathcal{H}_{\text{eff}} := \text{Span}\left\{ |k_1\rangle \otimes |k_2\rangle \otimes \cdots \otimes |k_n\rangle : \sum_{i=1}^{n} k_i \leq \left\lceil \frac{nE}{\varepsilon^2} \right\rceil \right\}$$

Let $\rho$ such that $\text{Tr}[\rho \hat{E}] \leq nE$ . Then, the projection $\rho_{\text{eff}}$ of $\rho$ onto $\mathcal{H}_{\text{eff}}$ satisfies

$$d_{\text{tr}}(\rho, \rho_{\text{eff}}) \leq \varepsilon$$

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# Mixed case

**Theorem**

Let $\rho$ be an **unknown** $n$ mode state with $\mathrm{Tr}[\rho\hat{E}] \leq nE$ . Then:

- There exists a tomography algorithm with $N = O\left(\frac{E^{2n}}{\varepsilon^{3n}}\right)$.
- Any tomography algorithm satisfies $N \geq \tilde{\Omega}\left(\frac{E^{2n}}{\varepsilon^{2n}}\right)$.

**Proof sketch of** "There is a tomography algorithm with $N = O\left(\frac{E^{2n}}{\varepsilon^{3n}}\right)$"

$$\dim \mathscr{H}_{\text{eff}} = \tilde{\Theta}\left(\left(\frac{E}{\varepsilon^2}\right)^n\right)$$

$$r_{\text{eff}} = \tilde{\Theta}\left(\left(\frac{E}{\varepsilon}\right)^n\right)$$

Hence, quantum state tomography is achievable with

$$N = \tilde{\Theta}\left(\frac{r_{\text{eff}} \dim \mathscr{H}_{\text{eff}}}{\varepsilon^2}\right) = \tilde{\Theta}\left(\frac{E^{2n}}{\varepsilon^{3n}}\right).$$

**Example 1**

$\mathcal{S} := \{\text{qu}D\text{it states}\}$ ⟶ $N(\mathcal{S}, \varepsilon, \delta) = \tilde{\Theta}\left(\frac{D^2}{\varepsilon^2}\right)$

**Example 2**

$\mathcal{S} := \{\text{qu}D\text{it pure states}\}$ ⟶ $N(\mathcal{S}, \varepsilon, \delta) = \tilde{\Theta}\left(\frac{D}{\varepsilon^2}\right)$

**Example 3**

$\mathcal{S} := \{\text{qu}D\text{it states with rank } r\}$ ⟶ $N(\mathcal{S}, \varepsilon, \delta) = \tilde{\Theta}\left(\frac{Dr}{\varepsilon^2}\right)$


[R. O'Donnell and J. Wright, Efficient quantum tomography (2015)]
[J. Haah, A. W. Harrow, Z. Ji, X. Wu, and N. Yu, Sample-optimal tomography of quantum states (2017)]
[R. Kueng, H. Rauhut, and U. Terstiege, Low rank matrix recovery from rank one measurements (2014)]

**Example 1**

$\mathcal{S} := \{\text{qu}D\text{it states}\}$ → $N(\mathcal{S}, \varepsilon, \delta) = \tilde{\Theta}\left(\frac{D^2}{\varepsilon^2}\right)$

**Example 2**

$\mathcal{S} := \{\text{qu}D\text{it pure states}\}$ → $N(\mathcal{S}, \varepsilon, \delta) = \tilde{\Theta}\left(\frac{D}{\varepsilon^2}\right)$

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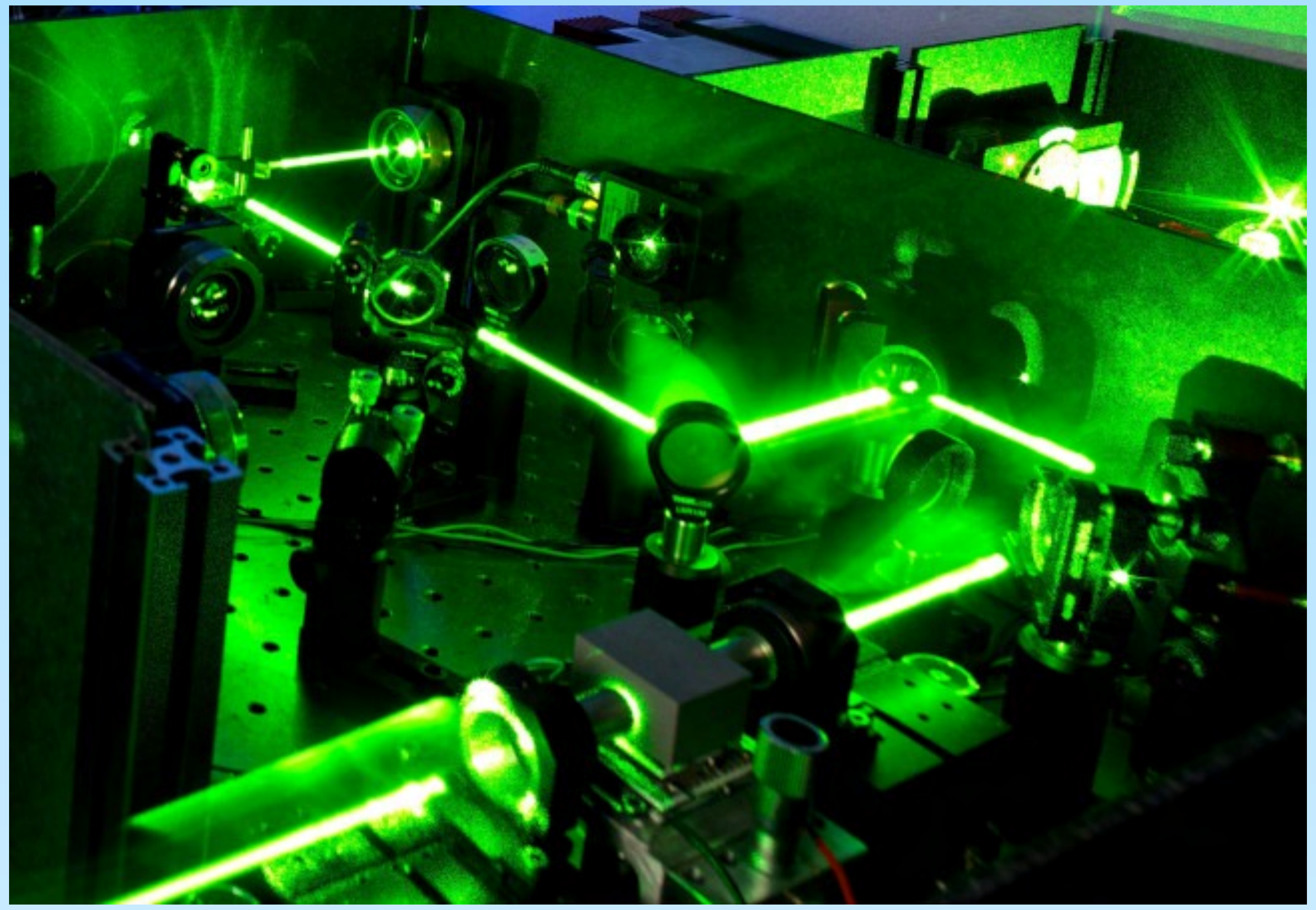
$n$-qubit states → $D = 2^n$ → Tomography is inefficient

“**Quantum state tomography**” of **optical systems**

Learning unknown states

Infinite-dimensional Hilbert spaces



The term “**tomography**” was first introduced in Quantum Physics in the context of **optical systems** [ Smithey, D. T. et al "*Measurement of the Wigner distribution and the density matrix of a light mode using optical homodyne tomography: Application to squeezed states and the vacuum*” (1993)]

Despite many (heuristic) approaches validated in quantum optics labs, the literature lacks “*rigorous performance guarantees*”

Our work fills this gap

We give guarantees wrt **trace distance**

# Improvement for pure Gaussian states

**Theorem**

Let $\psi$ be a pure Gaussian state, let $\tilde{\rho}$ be **any** state, with $\mathrm{Tr}[\psi\hat{E}], \mathrm{Tr}[\tilde{\rho}\hat{E}] \leq E_{\mathrm{tot}}$. Then:

$$d_{\mathrm{tr}}(\psi, \tilde{\rho}) \leq \sqrt{E_{\mathrm{tot}} + \frac{n}{2}} \left(2\|\mathrm{m}(\psi) - \mathrm{m}(\tilde{\rho})\|_2^2 + \|V(\psi) - V(\tilde{\rho})\|_\infty\right)^{1/2}$$

**Theorem**

Let $\rho$ be an **unknown** $n$ mode pure Gaussian state with $\mathrm{Tr}[\rho\hat{E}] \leq nE$. Then, a number

$$N = O\left(\frac{n^5 E^4}{\varepsilon^4} \log\left(\frac{n^2}{\delta}\right)\right)$$

of state copies suffices to output $\tilde{\rho}$ such that $\Pr\left[d_{\mathrm{tr}}(\tilde{\rho}, \rho) < \varepsilon\right] > 1 - \delta$

# Proof idea of the sufficient condition:

**Lemma**

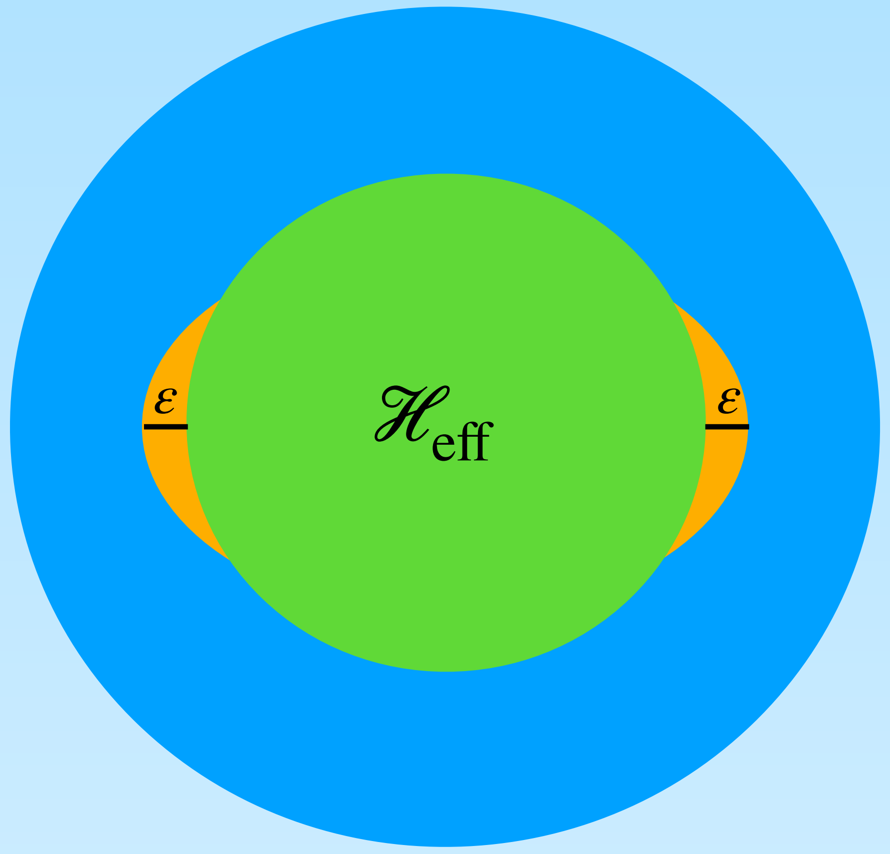
$$\mathscr{H}_{\mathrm{eff}} := \mathrm{Span}\left\{ |k_1\rangle \otimes |k_2\rangle \otimes \cdots \otimes |k_n\rangle : \sum_{i=1}^{n} k_i \leq \left\lceil \frac{nE}{\varepsilon^2} \right\rceil \right\}$$

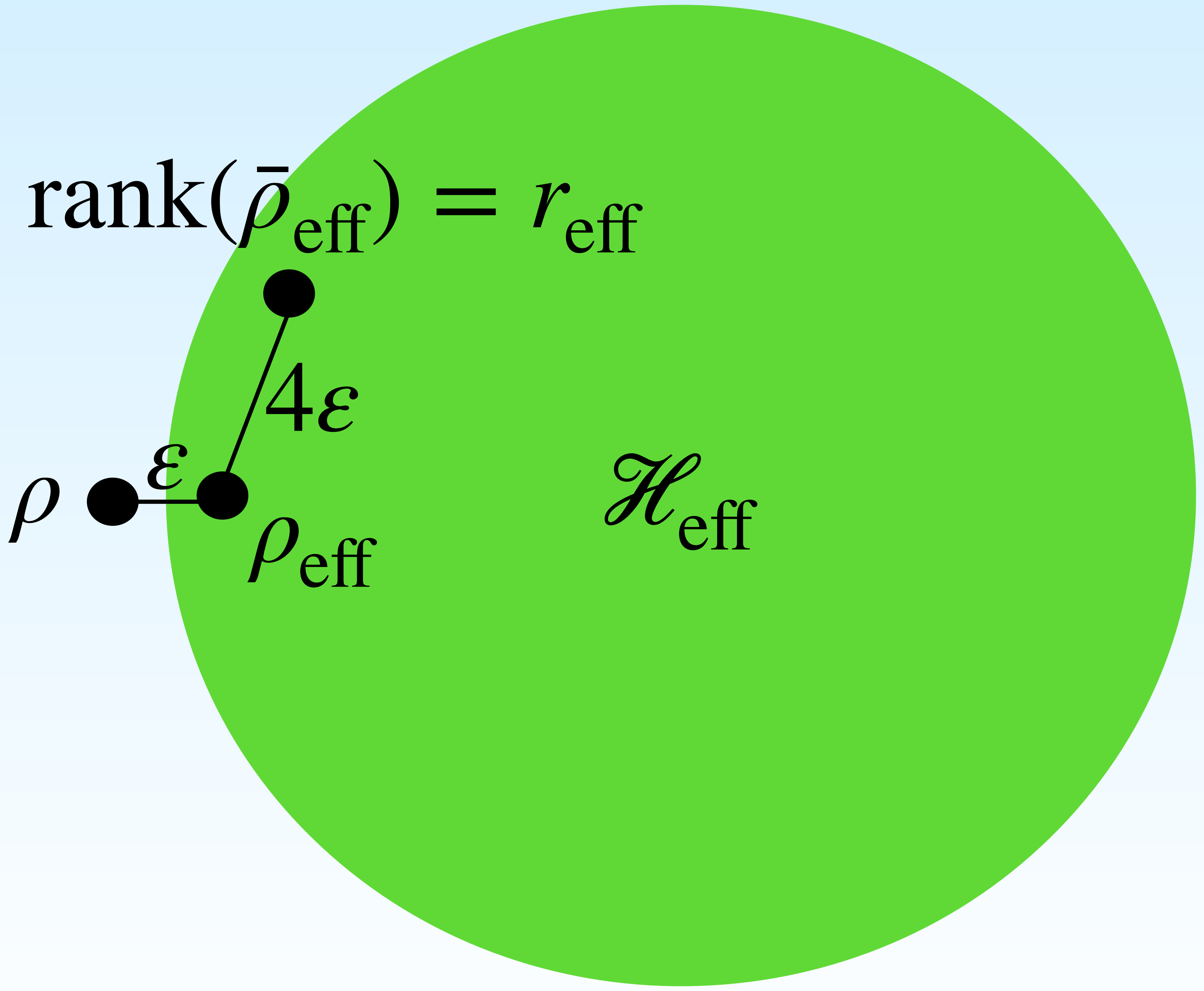
Let $\rho$ such that $\mathrm{Tr}[\rho \hat{E}] \leq nE$ . Then, the projection $\rho_{\mathrm{eff}}$ of $\rho$ onto $\mathscr{H}_{\mathrm{eff}}$ satisfies

$$d_{\mathrm{tr}}(\rho, \rho_{\mathrm{eff}}) \leq \varepsilon.$$

In addition, $\rho_{\mathrm{eff}}$ is $4\varepsilon$-close to a state of $\mathscr{H}_{\mathrm{eff}}$ with rank

$$r_{\mathrm{eff}} = \tilde{\Theta}\left( \left( \frac{E}{\varepsilon} \right)^n \right)$$



## Fundamental question

If we estimate $\mathrm{m}(\rho)$ and $V(\rho)$ of an unknown Gaussian state $\rho$ with precision $\varepsilon$, what is the resulting **trace distance error** on the state?

## Theorem

Let $\rho, \tilde{\rho}$ be Gaussian states with $\mathrm{Tr}[\rho\hat{E}], \mathrm{Tr}[\tilde{\rho}\hat{E}] \leq E_{\mathrm{tot}}$. Then:

$$d_{\mathrm{tr}}(\rho, \tilde{\rho}) \leq \sqrt{2(E_{\mathrm{tot}} + 1)} \left( \|\mathrm{m}(\rho) - \mathrm{m}(\tilde{\rho})\|_2 + \sqrt{2\|V(\rho) - V(\tilde{\rho})\|_1} \right)$$

**Not** proved via fidelity

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**Not** proved via fidelity

$$\mathcal{F}(\hat{\rho}_1, \hat{\rho}_2) = \mathcal{F}_0(V_1, V_2) \exp\left[-\frac{1}{4}\delta_u^T (V_1 + V_2)^{-1} \delta_u\right]$$

$$\mathcal{F}_0(V_1, V_2) = \frac{F_{\mathrm{tot}}}{\sqrt[4]{\det[V_1 + V_2]}},$$

$$F_{\mathrm{tot}}^4 = \det\left[2\left(\sqrt{\mathbb{1} + \frac{(V_{\mathrm{aux}}\Omega)^{-2}}{4}} + \mathbb{1}\right) V_{\mathrm{aux}}\right]$$

$$V_{\mathrm{aux}} = \Omega^T (V_1 + V_2)^{-1} \left(\frac{\Omega}{4} + V_2 \Omega V_1\right)$$

# Notation

Hilbert space of one **mode** $= \mathrm{Span}\{\,|0\rangle\,, |1\rangle, \cdots, |d\rangle, |d+1\rangle, \cdots\}$

Vacuum state
(0 photons)

($d$ photons)

$$\hat{a} := \sum_{k=1}^{\infty} \sqrt{k}\,|k-1\rangle\langle k|$$ (**annihilation** operator)

$$\hat{x} := \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}$$ (**position** operator)

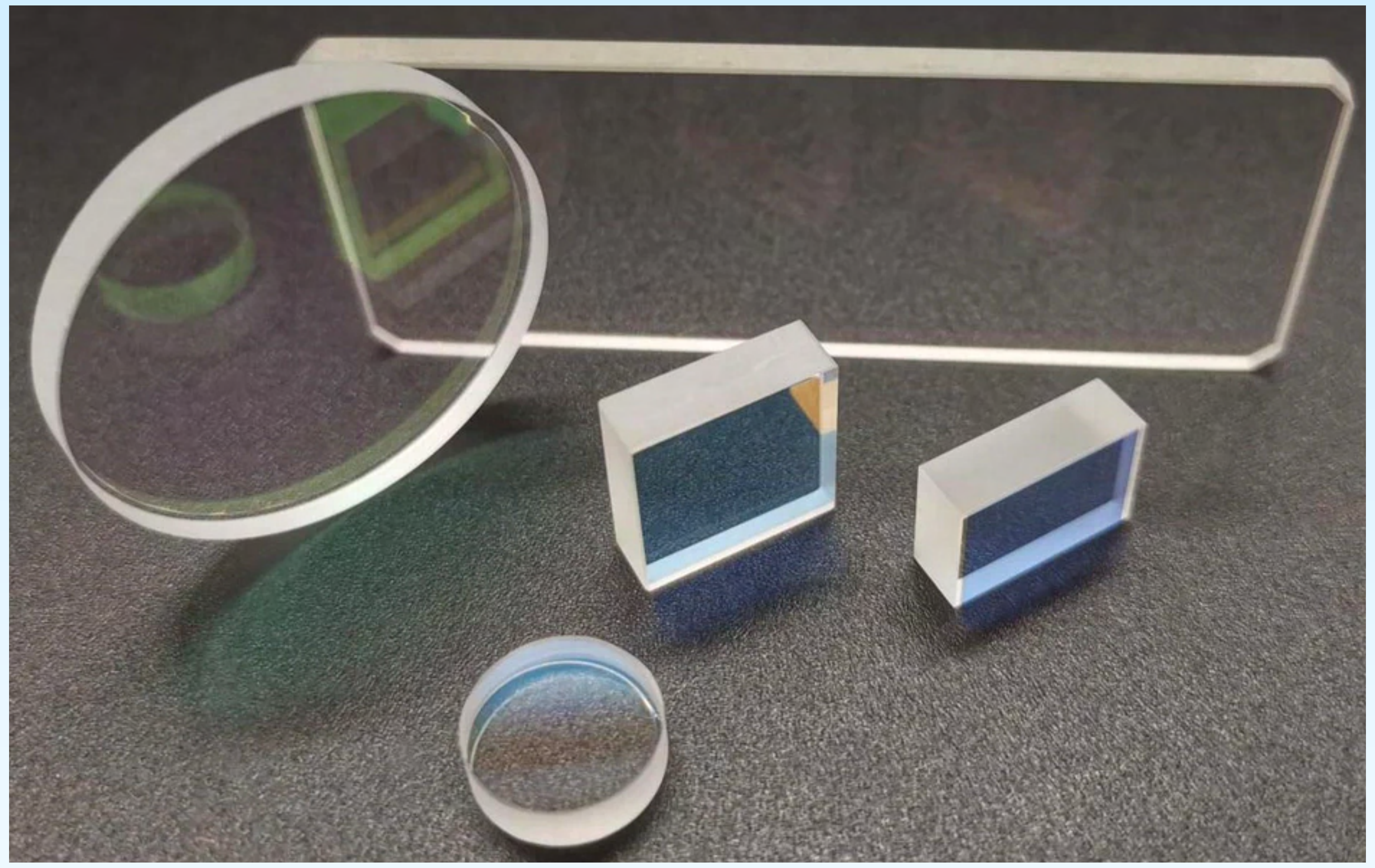
$$\hat{p} := \frac{\hat{a} - \hat{a}^\dagger}{\sqrt{2}i}$$ (**momentum** operator)

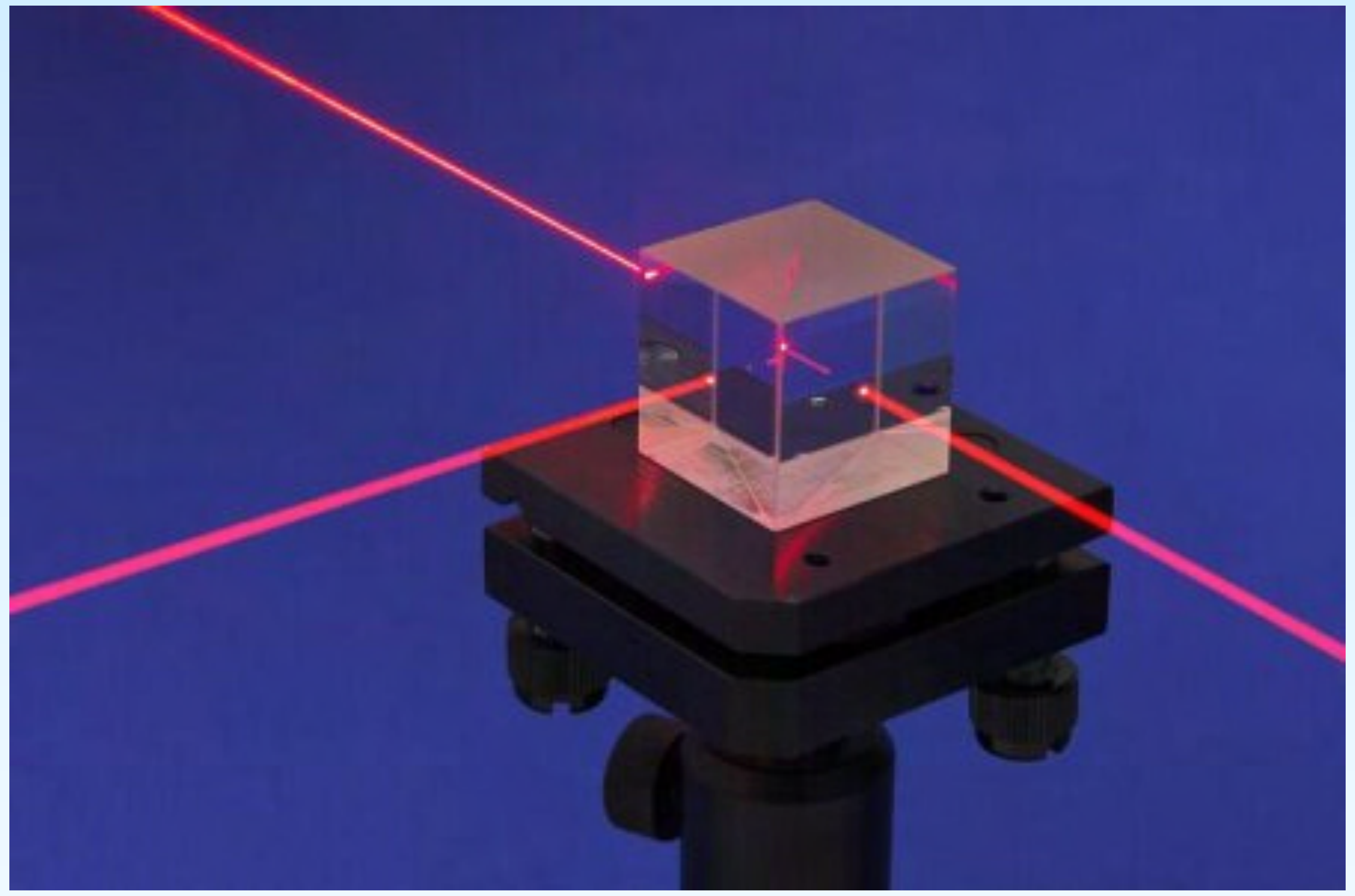
**Definition**

$\hat{G}$ is a **Gaussian unitary** if it is a composition of $e^{-i\hat{H}_{\text{quad}}}$, where $\hat{H}_{\text{quad}}$ is a (quadratic) Hamiltonian:

$$\hat{H}_{\text{quad}} := (\hat{\mathsf{R}} - \mathsf{m})^{\mathsf{t}}\, h\, (\hat{\mathsf{R}} - \mathsf{m}),$$

where $h \in \mathbb{R}^{2n,2n}$ is symmetric and $m \in \mathbb{R}^{2n}$.





Any pure Gaussian state $|\psi\rangle$ is of the form $|\psi\rangle = \hat{G}|0\rangle$, for some Gaussian unitary $\hat{G}$